# A PROBABILISTIC INTERPRETATION OF EULERIAN NUMBERS

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**1. Introduction.** The Eulerian numbers, denoted here by $\alpha_{n,k}$, may be defined by the recurrence

$$\alpha_{n+1,k} = (n - k + 2)\alpha_{n,k-1} + k\alpha_{n,k}$$

with the boundary conditions

$$\alpha_{0,0} = 1, \alpha_{n,0} = 0, \qquad n > 0.$$

An explicit formula for $\alpha_{n,k}$ is

$$\alpha_{n,k} = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k - j)^n,$$

which can be easily verified using the above recurrence.

Eulerian numbers, named after Euler who first discussed them in his famous book *Institutiones calculi differentialis* (St. Petersburg, 1755), arise in many varied contexts and have alternately been neglected and rediscovered for more than a century. The literature of classical combinatorial theory is replete with references to these numbers (often in varying guises) relating them to Stirling, tangent and Bernoulli numbers [3], [7], [11] as well as to the evaluation of certain infinite sums [9], [14].

The number $\alpha_{n,k}$ can be given a combinatorial interpretation in terms of permutations. A permutation $(\sigma_1, \sigma_2, \cdots, \sigma_n)$ of $(1, 2, \cdots, n)$ is said to have a *rise* at $\sigma_i$ if $\sigma_i < \sigma_{i+1}$; counting a conventional rise to the left of $\sigma_1$ it can be easily shown that $\alpha_{n,k}$ is the number of permutations of $(1, 2, \cdots, n)$ with exactly $k$ rises. Dual to the notion of rise is that of (ascending) *run*; placing a vertical line at both ends of $(\sigma_1, \cdots, \sigma_n)$ as well as between $\sigma_i$ and $\sigma_{i+1}$ whenever $\sigma_i > \sigma_{i+1}$, we see that the runs are the segments of the permutations between the pairs of lines [10]. It follows that $\alpha_{n,k}$ is the number of permutations with exactly $k$ ascending runs.

In this paper we shall be concerned primarily with the development of various asymptotic properties of Eulerian numbers. To date, the combinatorial interpretations given above have shed no light on these properties, probabilistic methods being used exclusively. It is precisely this fact which motivates the need for a more purely probabilistic interpretation of Eulerian numbers to which we now turn.

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