ON THE DISTRIBUTIONS OF CONJUGATE FUNCTIONS OF NONNEGATIVE MEASURES

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1. Introduction. In this paper properties of the distributions of conjugate functions $\tilde{f}$ of nonnegative functions f are interpreted in terms of Brownian motion using P. Levy's result that Brownian motion is a conformal invariant. It is shown that in the theorem of Kolmogorov $m(|\tilde{f}| > \lambda) < A \int |f| \, dm/\lambda$, $\lambda > 0$, the smallest possible constant A for nonnegative functions f is 1. Best possible one-sided Kolmogorov type inequalities are found, and related results are proved. The Hilbert transform is also considered. S. K. Pichorides in [5] has found the best constants in some strong type inequalities for conjugate functions and Hilbert transforms.

Few changes are required to make the proof of any result given here for the conjugate function work for the Hilbert transform, but the proofs for the Hilbert transform are more cluttered. Accordingly, in Section 2 only the conjugate function is considered, and in the last section the Hilbert transform is treated briefly.

Let D be the open unit disc in the complex plane, A be the unit circle $\{e^{i\theta}, 0 \leq \theta < 2\pi\}$ and H be the half plane $\{z : \text{Re } z > 0\}$. Let m be Lebesgue measure on A divided by 2π. If μ is a finite nonnegative Borel measure on A, $F = F_\mu$ will be defined on A by $g + i\tilde{\mu}$, where $g = d\mu/dm$ (the Radon-Nikodyn derivative of the nonsingular part of μ) and $\tilde{\mu}$ is the conjugate function of μ, and F will be defined in D by

$$F(z) = \int_A P(z, e^{i\theta}) \, d\mu(e^{i\theta}) + i \int_A Q(z, e^{i\theta}) \, d\mu(e^{i\theta})$$

where P and Q are the Poisson and Poisson conjugate kernels respectively. F is analytic on D and satisfies

$$\lim_{r \to 1} F(re^{i\theta}) = F(e^{i\theta}) \text{ a.e.} \tag{1}$$

with respect to m. (See Zygmund [8; Chapter 7].) Also $F(0) = \mu(A)$. We will always assume that μ is not a multiple of m so that F is not constant on D and

$$\text{Re } F(z) > 0, \qquad z \in D. \tag{2}$$

Let Z_t, $0 \leq t < \infty$, be two-dimensional Brownian motion in the complex plane such that $P(Z_0 = 0) = 1$, and let $\tau = \inf\{t : |Z_t| = 1\}$. Then the con-

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