SOME LOCALIZATIONS OF THE SPECTRAL MAPPING THEOREM

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One of the most fundamental results in the spectral theory of bounded linear operators on a complex Banach space X is the spectral mapping theorem due to Nelson Dunford [10]. This theorem (see also [11; Chapter VII, 3.11]) asserts that if T is a bounded linear operator in X and if f is a complex-valued function holomorphic on a neighborhood of the spectrum $\sigma(T)$ of T, then the spectrum of the operator $f(T)$ (defined by the contour integral $f(T) \equiv (1/2\pi i) \int_\Gamma f(\lambda)R(\lambda; T)\, d\lambda$, where Γ is a "scroc" enclosing $\sigma(T)$ and contained in the domain of f) is given by $\sigma(f(T)) = f(\sigma(T))$.

The purpose of this paper is to present several extensions of this basic theorem. We call them "localizations" of this result because they deal with the spectra of T and $f(T)$ "localized" to vectors x in X, with the spectra of the restrictions (or quotients) of T and $f(T)$ "localized" to certain subspaces, or with the spectra of the restrictions (or quotients) of T and $f(T)$ "localized" to vectors in certain subspaces.

A number of the main results are taken from the second author's dissertation [13] completed in 1966 at the University of Illinois.

1. The local spectral mapping theorem. In the following, X always denotes a complex Banach space and $B(X)$ denotes the algebra of all bounded linear operators in X. In the main, the terminology and notation will be that of [11] or [6], except that the word "subspace" always means a *closed* linear manifold.

We recall [6; p. 1], [11; p. 1931] that an operator $T \in B(X)$ is said to have the *single-valued extension property* if for any function f holomorphic on an open subset U of the complex plane $\mathbf{C}$ with values in X and such that $(\lambda I - T)f(\lambda) = 0$ for $\lambda \in U$, it follows that $f(\lambda) = 0$ for $\lambda \in U$. If T has this property and $x \in X$, then we define the *local resolvent set of* x (with respect to T) to be the set $\rho_T(x)$ of all $\lambda_0 \in \mathbf{C}$ such that there exists a function $\tilde{x}_T$ holomorphic on a neighborhood of λ_0 with values in X such that $(\lambda I - T)\tilde{x}_T(\lambda) = x$ for all λ. Since T has the single-valued extension property, such a function $\tilde{x}_T$ is evidently unique. Moreover $\tilde{x}_T(\lambda) = R(\lambda; T)x$ for $\lambda \in \rho(T)$, the resolvent set of T, and so $\rho(T) \subseteq \rho_T(x)$. Clearly $\rho_T(x)$ is open so its complement $\sigma_T(x) \equiv \mathbf{C} - \rho_T(x)$, which is contained in $\sigma(T)$, is compact. We call $\sigma_T(x)$ the *local spectrum of* x (with respect to T). If $F \subseteq \mathbf{C}$ is closed, we introduce the linear manifold

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