BERRY-ESSEEN BOUNDS AND A THEOREM OF ERDÖS AND TURÁN ON UNIFORM DISTRIBUTION MOD 1

H. NIEDERREITER AND WALTER PHILIPP

1. Introduction. Let $\langle x_n \rangle$, $n = 1, 2, \cdots$, be a sequence of real numbers contained in $[0, 1)$. Denote by $A(N, x)$ the number of $n \leq N$ with $x_n < x$. The sequence $\langle x_n \rangle$ is called uniformly distributed (u.d.) if $N^{-1}A(N, x) \to x$ as $N \to \infty$ for all $0 \leq x \leq 1$. (In general, $\langle x_n \rangle$ is called u.d. mod 1 if the sequence of fractional parts $\{x_n\}$ is u.d.) It is easy to see that $\langle x_n \rangle$ is u.d. if and only if

$$D_N^* \overset{\text{def}}{=} \sup_{0 \leq x \leq 1} |N^{-1}A(N, x) - x| \to 0.$$

Another equivalent condition is the Weyl criterion: $\langle x_n \rangle$ is u.d. if and only if

$$S_N(h) \overset{\text{def}}{\equiv} N^{-1} \sum_{n \leq N} e^{2\pi i h x_n} \to 0$$

for all $h \in \mathbf{Z} - \{0\}$. (For the proof of the basic theorems see [5].) The following theorem due to Erdös and Turán [3] can be regarded as a quantitative version of the sufficient part of the Weyl criterion.

THEOREM A. *For any integer* $m \geq 1$

$$D_N^* \leq c_1 \frac{1}{m+1} + c_2 \sum_{h=1}^{m} \frac{1}{h} |S_N(h)| \cdot$$

The best constants c_1 and c_2 so far have been $c_1 = 17.2$ and $c_2 = 4.3$ (Niederreiter, unpublished). Much larger values were given by Yudin [9].

The purpose of this paper is to give various generalizations of this theorem and to point out their connections with other parts of analysis. We shall prove the following theorem.

THEOREM 1. *Let* $F(x)$ *be nondecreasing on* $[0, 1]$ *with* $F(0) = 0$ *and* $F(1) = 1$, *and let* $G(x)$ *satisfy a Lipschitz condition on* $[0, 1]$, *i.e.*,

$$|G(x) - G(y)| \leq M\,|x - y|$$

for all $0 \leq x, y \leq 1$. *Suppose that* $G(0) = 0$ *and* $G(1) = 1$. *Then for any positive integer* m

$$\sup_{0 \leq x \leq 1} |F(x) - G(x)| \leq \frac{4M}{m+1} + \frac{4}{\pi} \sum_{h=1}^{m} \left(\frac{1}{h} - \frac{1}{m+1}\right) |\hat{F}(h) - \hat{G}(h)|$$

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