LINEAR HOMOGENEOUS DIOPHANTINE EQUATIONS AND MAGIC LABELINGS OF GRAPHS

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1. Introduction. Let G be a finite graph allowing loops and multiple edges. Hence G is a *pseudograph* in the terminology of [10]. We shall denote the set of vertices of G by V, the set of edges by E, the number $|V|$ of vertices by p, and the number $|E|$ of edges by q. Also if an edge e is incident to a vertex v, we write $v \in e$. Any undefined graph-theoretical terminology used here may be found in [10]. A *magic labeling of G of index r* is an assignment $L : E \rightarrow \{0, 1, 2, \cdots\}$ of a nonnegative integer $L(e)$ to each edge e of G such that for each vertex v of G the sum of the labels of all edges incident to v is r (counting each loop at v once only). In other words,

$$\sum_{e : v \in e} L(e) = r, \qquad \text{for all } v \in V. \tag{1}$$

For each edge e of G let z_e be an indeterminate and let z be an additional indeterminate. For each vertex v of G define the homogeneous linear form

$$P_v = z - \sum_{e : v \in e} z_e , \qquad v \in V, \tag{2}$$

where the sum is over all e incident to v. Hence by (1) a magic labeling L of G corresponds to a solution of the system of equations

$$P_v = 0, \qquad v \in V, \tag{3}$$

in nonnegative integers (the value of z is the index of L). Thus the theory of magic labelings can be put into the more general context of *linear homogeneous diophantine equations*. Many of our results will be given in this more general context and then specialized to magic labelings.

It may happen that there are edges e of G that are always labeled 0 in any magic labeling. If this is the case, then these edges may be ignored in so far as studying magic labelings is concerned; so we may assume without loss of generality that for any edge e of G there is a magic labeling L of G for which $L(e) > 0$. We then call G a *positive* pseudograph. If in a magic labeling L of G every edge receives a positive label, then we call L a *positive* magic labeling. If L_1 and L_2 are magic labelings, we define their *sum* $L = L_1 + L_2$ by $L(e) = L_1(e) + L_2(e)$ for every edge e of G. Clearly if L_1 and L_2 are of index r_1 and r_2, then L is magic of index $r_1 + r_2$. Now note that every positive pseudograph G possesses a positive magic labeling L, e.g., for each edge e of G let L_e be a magic labeling positive on e, and let $L = \sum L_e$.

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