COMPRESSION AND EXPANSION OF BOUNDARY SETS

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1. Introduction. Let f denote a univalent conformal mapping of the unit disk D onto a Jordan domain G, together with its continuous extension to the unit circle C. If the boundary Γ of G is sinuous, then f tends to expand portions of C that correspond to outward bulges of Γ and to compress portions of C that correspond to inward bulges of Γ. This relation underlies the following proposition.

THEOREM (M. Lavrent'ev [5; p. 830 and p. 18 in the translation]). *There exists a Jordan domain G such that every univalent conformal map from D onto G carries some set of positive measure on C onto a set of linear measure 0 on Γ.*

Because the original proof of Lavrent'ev's theorem [5; pp. 830–847] is difficult, we give a new proof (see Section 2). Our construction leads to a slightly stronger result: The boundary Γ of our domain G contains a set whose linear measure is 0 and whose image on C (under any conformal mapping of G onto D) has measure 2π; moreover, each univalent conformal mapping of D onto G belongs to the class (A) of functions whose power series converge absolutely on C.

In the other direction, A. J. Lohwater and W. Seidel [7] first demonstrated the phenomenon of violent expansion of boundary sets by constructing a Jordan domain whose boundary meets a line segment in a set of positive Lebesgue measure and of harmonic measure 0 relative to the domain. In Section 3, we construct an example for which at least one mapping function belongs to the class (A).

F. Bagemihl and G. Piranian [1; §2] have constructed functions that belong to the class (A), have a *finite Dirichlet integral*, and map the unit circle C onto a Peano curve. The question whether a *univalent* function of class (A) can map C onto a curve of positive two-dimensional Lebesgue measure remains open; but in Section 4 we answer it in a weaker form in which positive Lebesgue 2-measure is replaced with Hausdorff dimension 2.

2. Compression of boundary sets.

THEOREM 1. *There exists a Jordan domain G such that every univalent conformal map of the unit disk onto G belongs to the class (A) and carries a set of measure 2π on the unit circle C onto a set of linear measure 0.*

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