REDUCING ESSENTIAL EIGENVALUES

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1. Introduction. Throughout this paper $\mathcal{H}$ will denote a fixed separable, infinite dimensional complex Hilbert space and $\mathcal{B}(\mathcal{H})$ will denote the algebra of all (bounded, linear) operators on $\mathcal{H}$. The (uniformly closed two-sided) ideal of all compact operators on $\mathcal{H}$ will be denoted by $\mathcal{C}(\mathcal{H})$ and the canonical quotient map from $\mathcal{B}(\mathcal{H})$ onto the (Calkin) algebra $\mathcal{A}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{C}(\mathcal{H})$ will be denoted by π. An eigenvalue λ of an operator T in $\mathcal{B}(\mathcal{H})$ is called a *reducing eigenvalue* of T if the eigenspaces of T and T^* corresponding to λ and $\bar{\lambda}$ respectively have nontrivial intersection or, equivalently, if null $(T - \lambda) \cap$ null $(T^* - \bar{\lambda}) \neq \{0\}$. (Observe that null $(T - \lambda) \cap$ null $(T^* - \bar{\lambda})$ reduces T.) If λ is a reducing eigenvalue of T and in addition null $(T - \lambda) =$ null $(T^* - \bar{\lambda})$, then λ will be called a *normal eigenvalue* of the operator T. The notions of reducing eigenvalue and normal eigenvalue coincide with the notion of eigenvalue when we restrict ourselves to the class of normal operators. However, it is of interest to inquire for which complex numbers λ the notions mentioned above coincide when the operator under discussion is not necessarily normal. This problem has received a great deal of attention in the past years. We give a brief and incomplete account of the results in the subject in Section 2 (only those facts which we need are mentioned).

It is our purpose in the present paper to introduce the notions of reducing eigenvalues and normal eigenvalues for elements in the Calkin algebra $\mathcal{A}(\mathcal{H})$ by analogy with the concepts of reducing eigenvalues and normal eigenvalues for an operator in $\mathcal{B}(\mathcal{H})$. Note that λ is a reducing eigenvalue of an operator T in $\mathcal{B}(\mathcal{H})$ if and only if there exists a nonzero (orthogonal) projection P in $\mathcal{B}(\mathcal{H})$ such that $(T - \lambda)P = (T^* - \bar{\lambda})P = 0$. Likewise, λ is a normal eigenvalue of T if and only if it is a reducing eigenvalue and for every projection Q in $\mathcal{B}(\mathcal{H})$ we have $(T - \lambda)Q = 0$ if and only if $(T^* - \bar{\lambda})Q = 0$. We use these last descriptions of reducing eigenvalues and normal eigenvalues to define the above mentioned analogous concepts in $\mathcal{A}(\mathcal{H})$.

Definition. We shall say that a complex number λ is a reducing essential eigenvalue of an operator T in $\mathcal{B}(\mathcal{H})$ if there exists a projection P in $\mathcal{B}(\mathcal{H})$ such that $\pi(P) \neq 0$ and $\pi(T - \lambda)\pi(P) = \pi(T^* - \bar{\lambda})\pi(P) = 0$. If λ is a reducing essential eigenvalue of T and, in addition, for each projection Q in $\mathcal{B}(\mathcal{H})$ we have $\pi(T - \lambda)\pi(Q) = 0$ if and only if $\pi(T^* - \bar{\lambda})\pi(Q) = 0$, then we shall say that λ is a normal essential eigenvalue of T.

The set of all reducing essential eigenvalues of an operator T will be denoted

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