ON THE DISTORTION OF SETS ON A JORDAN CURVE UNDER CONFORMAL MAPPING

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1. Let J be a Jordan curve and let Ω denote the exterior of J. If $E \subset J$, we denote by $\omega(E)$ the harmonic measure of E with respect to Ω and ∞. We denote by $\Lambda_h(E)$ the Hausdorff measure of E with respect to some continuous increasing function $h(t)$, i.e.,

$$\Lambda_h(E) = \lim_{\epsilon \to 0} \sum h(r_i),$$

where r_i are the radii of discs C_i covering E and $r_i \leq \epsilon$. Our problem concerns the possible implications $\Lambda_h(E) = 0 \Rightarrow \omega(E) = 0$. Two facts are known. Lavrentiev [1] has proved that $\Lambda_h(E) = 0 \not\Rightarrow \omega(E) = 0$ if $h(r) = r$. Conversely, it follows immediately from the Beurling projection theorem for harmonic measures that $\Lambda_h(E) = 0 \Rightarrow \omega(E) = 0$ if $h(r) = r^{\frac{1}{2}}$. The proof in [1] is very complicated and McMillan–Piranian [2] have given a simple example.

The purpose of this note is to show that the problem is to a large extent still not solved and to indicate some new tools. First we give a new example which is in a sense canonical and also gives a result on which h-functions are possible. It is possible that our examples come close to the true best possible result. As a small step in this direction we show that $\frac{1}{2}$ in Beurling's result can be increased.

THEOREM. (A) *If*

$$h(r) = r \exp\left\{\left(\log \frac{1}{r}\right)^{\alpha}\right\}, \qquad 0 < \alpha < \tfrac{1}{2},$$

there exist J and $E \subset J$ with $\Lambda_h(E) = 0$ and $\omega(E) > 0$.

(B) *There is a number β, $\beta > \frac{1}{2}$, so that if $\Lambda_h(E) = 0$ and $h(r) = r^{\beta}$, then $\omega(E) = 0$.*

2. Let γ be a simple, smooth curve and suppose that $1 \in \gamma$ and that $0 \in \operatorname{int}(\gamma)$. We also assume that $\operatorname{int}(\gamma)$ is starshaped with respect to 0. Let n be a (large) integer and let Γ_n be the curve $\{z \mid z^n \in \gamma\}$. Γ_n can be written in polar coordinates as

$$\Gamma_n : r = 1 + \varphi_n\left(\frac{\theta}{2\pi}\right).$$

Our curve J is constructed as follows. Let J_0 be a simple, smooth Jordan

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