ON THE q-ANALOG OF KUMMER'S THEOREM AND APPLICATIONS

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1. Introduction. The q-analogs for Gauss's summation of ${}_2F_1[a, b; c; 1]$ and Saalschutz's summation of ${}_3F_2[a, b, -n; c, a+b-c-n+1; 1]$ are well known, namely, E. Heine [8; p. 107, Equation (6)] showed that

$$(1.1)\qquad {}_2\phi_1\begin{bmatrix} a,\ b;\ q,\ c/ab \\ c \end{bmatrix} = \frac{(c/a)_\infty (c/b)_\infty}{(c)_\infty (c/ab)_\infty}$$

where

$${}_m\phi_n\begin{bmatrix} a_1\ ,\ \cdots\ ,\ a_m\ ;\ q,\ z \\ b_1\ ,\ \cdots\ ,\ b_n \end{bmatrix} = \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_m)_j z^j}{(q)_j (b_1)_j \cdots (b_n)_j}\ ,$$

and $(a)_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$, $(a)_\infty = (a;q)_\infty = \lim_{n\to\infty}(a)_n$. (See also [12; p. 97, Equation (3.3.2.2)].) F. H. Jackson [9; p. 145] showed that

$$(1.2)\qquad {}_3\phi_2\begin{bmatrix} a,\ b,\ q^{-n};\ q,\ q \\ c,\ abq/cq^n \end{bmatrix} = \frac{(c/a)_n (c/b)_n}{(c)_n (c/ab)_n}\ .$$

The q-analog of Dixon's summation of ${}_3F_2[a, b, c; 1+a-b, 1+a-c; 1]$ was more difficult to find, and indeed only a partial analog is true; namely, W. N. Bailey [5] and F. H. Jackson [10; p. 167, Equation (2)] proved that if $a = q^{-2n}$ where n is a positive integer, then

$$(1.3)\qquad {}_3\phi_2\begin{bmatrix} a,\ b,\ c;\ q,\ \dfrac{q^2a^{\frac12}}{bc} \\ \dfrac{aq}{b}\ ,\ \dfrac{aq}{c} \end{bmatrix} = \frac{(b/a)_\infty (c/a)_\infty (qa^{\frac12})_\infty (bca^{-\frac12})_\infty}{(ba^{-\frac12})_\infty (ca^{-\frac12})_\infty (a^{-1}qa)_\infty (bca^{-1})_\infty}\ .$$

There are three other well-known summations for the ${}_2F_1$ series, namely, Kummer's theorem [12; p. 243, Equation (III. 5)]

$$(1.4)\qquad {}_2F_1[a, b; 1+a-b; -1] = \frac{\Gamma(1+a-b)\Gamma\left(1+\frac{a}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{a}{2}\right)\Gamma\left(1+\frac{a}{2}-b\right)}\ ,$$

Gauss's second theorem [12; p. 243, Equation III. 6)]

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