BASIC INTERVALS AND RELATED SUBLATTICES OF THE LATTICE OF TOPOLOGIES

RICHARD VALENT

1. Introduction. In [5] R. E. Larson and the present author considered questions concerning the lattice structure of basic intervals in the lattice of topologies. In this article we continue this investigation. We shall adhere to the notation of [5], namely, X is a fixed set, $\mathcal{R}$, $\mathcal{S}$, and $\mathfrak{I}$ are topologies on X, Σ is the lattice of topologies on X, and Λ is the sublattice of T_1-topologies on X.

At the outset, we modify the definition of basic interval to include a slightly wider class of intervals. Whereas the definition given in [5] is adequate for studying intervals of T_1-topologies, the definition given here is more appropriate for studying intervals of arbitrary topologies.

In the next section we use the class of basic intervals of Σ to construct certain sublattices of Σ having desirable lattice properties. As a special case we obtain the power sublattices of Λ introduced by Larson and Thron in [4]. The method used in the construction of these power sublattices is shown to apply to T_0-topologies, and an adaptation is given for non-T_0-topologies as well. In the course of this, a somewhat surprising fact comes to light: Every finite interval of Λ is distributive.

In [5] it is shown that every basic interval is complete, infinitely meet distributive, and compactly generated. We give an example in the final section of this paper which establishes that not every complete, infinitely meet distributive, compactly generated lattice is isomorphic to some basic interval.

We thank the referee for his many suggestions and improvements.

2. Basic and very basic intervals. Let $\mathcal{S}, \mathfrak{I} \in \Sigma$ and $x \in X$. As usual, $\mathfrak{N}(\mathcal{S}, x)$ and $\mathfrak{N}(\mathfrak{I}, x)$ denote the neighborhood filters of $\mathcal{S}$ and $\mathfrak{I}$, respectively, at x. In [2] Banaschewski defines $\mathcal{S}$ to be an *expansion of* $\mathfrak{I}$ *by* a, where $a \in X$, if and only if $\mathfrak{N}(\mathcal{S}, a)$ is properly contained in $\mathfrak{N}(\mathfrak{I}, a)$ and $\mathfrak{N}(\mathcal{S}, x) = \mathfrak{N}(\mathfrak{I}, x)$ for all $x \in X - \{a\}$.

THEOREM 1. *Let $\mathcal{S}, \mathfrak{I} \in \Sigma$ be given such that $\mathcal{S} < \mathfrak{I}$. $\mathcal{S}$ is an expansion of $\mathfrak{I}$ by a if and only if for all $G \in \mathfrak{I} - \mathcal{S}$, $a \in G$ and $G - \{a\} \in \mathcal{S}$.*

Proof. Suppose $\mathcal{S}$ is an expansion of $\mathfrak{I}$ by a, and $G \in \mathfrak{I} - \mathcal{S}$. Then $G \in \mathfrak{N}(\mathfrak{I}, a)$ so $a \in G$. Since $\mathfrak{N}(\mathcal{S}, x) = \mathfrak{N}(\mathfrak{I}, x)$ for all $x \in X - \{a\}$, for each $x \in G - \{a\}$ there exists an $N_x \in \mathcal{S}$ such that $N_x \subseteq G$. Therefore $G - \{a\} \in \mathcal{S}$.

Next, suppose $a \in X$ satisfies $a \in G$ and $G - \{a\} \in \mathcal{S}$ for all $G \in \mathfrak{I} - \mathcal{S}$. Then $\mathfrak{N}(\mathcal{S}, x) = \mathfrak{N}(\mathfrak{I}, x)$ for all $x \in X - \{a\}$. Since $\mathcal{S} < \mathfrak{I}$, $\mathfrak{N}(\mathcal{S}, a)$ is properly contained in $\mathfrak{N}(\mathfrak{I}, a)$, and so $\mathcal{S}$ is an expansion of $\mathfrak{I}$ by a.

Received October 6, 1972. Revisions received April 25, 1973.