RING TOPOLOGIES ON THE QUOTIENT FIELD OF A DEDEKIND DOMAIN

J. HEINE AND SETH WARNER

Let D be a Dedekind domain that is not a field, and let K be the quotient field of D. A ring topology $\mathfrak{J}$ on K is *D-linear* if the open D-submodules form a fundamental system of neighborhoods of zero for $\mathfrak{J}$. Let P be the set of all nonzero prime ideals of D. We recall that the fractionary ideals of K are precisely the finitely generated D-submodules and that a nonzero fractionary ideal $\mathfrak{a}$ has the unique factorization

$$\mathfrak{a} = \prod_{\mathfrak{p}\in P} \mathfrak{p}^{n_{\mathfrak{p}}(\mathfrak{a})}$$

where $n_{\mathfrak{p}}(\mathfrak{a}) \in \mathbf{Z}$ and $n_{\mathfrak{p}}(\mathfrak{a}) = 0$ for all but finitely many $\mathfrak{p} \in P$. Moreover, if $\mathfrak{a}$ and $\mathfrak{b}$ are nonzero fractionary ideals, then

$$n_{\mathfrak{p}}(\mathfrak{a} + \mathfrak{b}) = \min \{n_{\mathfrak{p}}(\mathfrak{a}), n_{\mathfrak{p}}(\mathfrak{b})\},$$

$$n_{\mathfrak{p}}(\mathfrak{a}\mathfrak{b}) = n_{\mathfrak{p}}(\mathfrak{a}) + n_{\mathfrak{p}}(\mathfrak{b}),$$

and $\mathfrak{a} \subseteq \mathfrak{b}$ if and only if $n_{\mathfrak{p}}(\mathfrak{a}) \geq n_{\mathfrak{p}}(\mathfrak{b})$ for all $\mathfrak{p} \in P$ [4; pp. 25–26]. For each $\mathfrak{p} \in P$ we denote the $\mathfrak{p}$-adic valuation by $v_{\mathfrak{p}}$. Thus for each nonzero $x \in K$, $v_{\mathfrak{p}}(x) = n_{\mathfrak{p}}(Dx)$.

Our goal is to describe all nondiscrete D-linear ring topologies on K and to characterize certain classes of such topologies: those that are locally bounded, are field topologies, have a fundamental system of neighborhoods of zero consisting of subrings, etc. In particular, we shall obtain the Correl–Jebli theorem that a nondiscrete D-linear ring topology $\mathfrak{J}$ is a field topology if and only if $\mathfrak{J}$ is the supremum of $\mathfrak{p}$-adic topologies.

1. The D-linear ring topology determined by a submodule. A nonzero fractionary ideal $\mathfrak{a}$ is completely determined by the function $\mathfrak{p} \mapsto n_{\mathfrak{p}}(\mathfrak{a})$ from P into $\mathbf{Z}$, in that nonzero fractionary ideals $\mathfrak{a}$ and $\mathfrak{b}$ are identical if $n_{\mathfrak{p}}(\mathfrak{a}) = n_{\mathfrak{p}}(\mathfrak{b})$ for all $\mathfrak{p} \in P$. More generally, a nonzero D-submodule V of K is similarly determined by the function φ_V from P into $\mathbf{Z} \cup \{-\infty\}$ defined by

$$\varphi_V(\mathfrak{p}) = \inf \{v_{\mathfrak{p}}(x) : x \in V\}$$

[4; p. 91, Exercise 3]. For completeness we shall give a proof.

THEOREM 1. *Let $\mathfrak{M}(D)$ be the lattice of all nonzero D-submodules of K, ordered by $\subseteq$, and let $\mathfrak{F}(D)$ be the lattice of all functions f from P into $\mathbf{Z} \cup \{-\infty\}$ such that $f(\mathfrak{p}) \leq 0$ for all but finitely many $\mathfrak{p} \in P$, ordered by $\geq$. Then $V \mapsto \varphi_V$ is a*

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