EXPONENTIALS IN DIFFERENTIALLY ALGEBRAIC EXTENSION FIELDS

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The analogue in differential algebra of the theorem of primitive elements has been studied in [1] and [3]. Kolchin [3] has shown for a differential field (F, δ) of characteristic zero the existence of primitive elements in finite differentially algebraic extensions of F in the case where the field F has an element f such that $\delta f \neq 0$. In [1] we constructed primitive elements in finite logarithmic differential extension fields. A logarithmic differential field is a differential field $(F\langle x_0, \cdots, x_m\rangle, \delta)$ such that $\delta f = 0$ for every $f \in F$, $\delta x_0 = 1$, and $x_{k-1} \cdots x_0 \delta x_k = 1$ for $0 < k \leq m$. These differential fields play an essential role in the asymptotic theory of ordinary differential equations. In this paper we consider extensions of differential fields by exponentials. Given a differential field (K, δ) with subfield of constants C we say the element $\beta \in K$ is *exponential* if $\beta \neq 0$ and $\delta\beta = c\beta$ for some nonzero $c \in C$. In the terminology of Kolchin [4; §23] β is an exponential of an integral of c. The differential field (K, δ) is called *exponential-free* if it has no exponentials. A sequence $\beta_1, \beta_2, \cdots$ in (K, δ) is a *tower of exponentials* if β_1 is an exponential of (K, δ) and β_n is an exponential of $(K, (\beta_1 \cdots \beta_{n-1})^{-1}\delta)$ for $n = 2, \cdots$. We show that if (F, δ) is exponential-free and $(F\langle\beta_1, \cdots, \beta_n\rangle, \delta)$ is an extension by a tower of exponentials, then $\beta_1, \beta_2, \cdots, \beta_n$ are algebraically independent over F and $(F\langle\beta_1, \cdots, \beta_n\rangle, \delta) = (F\langle\beta_n\rangle, \delta)$. We apply our results on extensions by exponentials to the theory of differential equations. We show there exists no differential equation of order less than n with coefficients in the exponential-free differential field (F, δ) whose solution is $\beta_n \in (F\langle\beta_1, \cdots, \beta_n\rangle, \delta)$, but there exists a differential equation of order n satisfied by β_n (e.g., there is no differential equation of order less than n with coefficients in $(R\langle x, \log x, \cdots, \log_m x\rangle, D)$ whose solution is $\exp_n x$ but there is a differential equation of order n with constant coefficients whose solution is $\exp_n x$, where $\log_1 x = \log x$, $\log_k x = \log(\log_{k-1} x)$ and similarly $\exp_1 x = \exp x$, $\exp_k x = \exp(\exp_{k-1} x)$, R is the reals, x is a real variable and D is the usual derivation of functions of one real variable).

All differential fields considered here are ordinary, are of characteristic zero, and are contained in a fixed differential field (K, δ). The subfield of constants of K will be denoted by C.

LEMMA 1. *Let (F_1, δ) be a differential extension field of the exponential-free differential field (F_0, δ). If $\alpha \in (F_1, \delta)$ is exponential, then*

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