FLATNESS CRITERIA FOR MODULES OF HOLOMORPHIC FUNCTIONS OVER $\mathcal{O}_n$

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1. Introduction. Let $D \subset \mathbf{C}^n$ be open, and suppose that the origin $0 \in \mathbf{C}^n$ is a boundary point of D. Let $\mathcal{O}_n$ denote the ring of germs of holomorphic functions at 0 in $\mathbf{C}^n$, and let $\mathcal{O}_D$ denote the ring of "germs", at 0, of holomorphic functions on D. ($\mathcal{O}_D$ is defined precisely as follows. For any open set $U \subset \mathbf{C}^n$ if $\mathcal{O}(U)$ is the ring of holomorphic functions on U and if $U_1 \subset U_2$, there is a natural restriction map $\mathcal{O}(U_2) \to \mathcal{O}(U_1)$. Then

$$\mathcal{O}_D = \text{dir lim } \mathcal{O}(D \cap U),$$

where the direct limit is taken over all open neighborhoods U of 0.)

Let M be a sub-$\mathcal{O}_n$-module of $\mathcal{O}_D$. The object of this paper is to determine, in certain special cases, whether M is a flat $\mathcal{O}_n$-module in the algebraic sense. Such submodules can arise by considering the set of germs in $\mathcal{O}_D$ which satisfy a given type of boundary behavior near the origin. In this paper we deal primarily with the following special examples, although some of our results hold more generally.

1. If m is a non-negative integer or $+\infty$ and if α is a real number, $0 \leq \alpha \leq 1$, let $M = \mathcal{W}_{m,\alpha}$ denote the ring of those germs of Whitney C^m-functions on $\bar{D}$, the closure of D, at 0 such that every m-th derivative satisfies a Hölder condition of order α. If $m = +\infty$, we shall suppress the α. $\mathcal{W}_{m,\alpha}$ is a ring extension of $\mathcal{O}_n$.

2. If $1 \leq p \leq +\infty$, let $M = \mathcal{L}_p$ denote the submodule of $\mathcal{O}_D$ of germs of holomorphic functions which are also germs of L^p functions on D. Thus if $U \subset \mathbf{C}^n$ is open and if $L^p(U)$ denotes the usual L^p-space of functions on U with respect to Lebesgue measure on $\mathbf{C}^n$, then

$$\mathcal{L}_p = \text{dir lim } [\mathcal{O}(D \cap U) \cap L^p(D \cap U)],$$

where the direct limit is taken over all open neighborhoods U of 0 in $\mathbf{C}^n$. If $p = +\infty$, then $\mathcal{L}_\infty$ is a ring extension of $\mathcal{O}_n$, but if $p < +\infty$, then $\mathcal{L}_p$ is only a sub-$\mathcal{O}_n$-module of $\mathcal{O}_D$.

The question of whether a given type of boundary behavior determines a flat $\mathcal{O}_n$-module M is important if, for example, it is known that the sheaf of germs of holomorphic functions on the domain D which satisfy the given boundary behavior at every boundary point has vanishing higher cohomology. It is then possible to pass from the local information about flatness of the module of germs of functions to global information about the space of holomorphic

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