EXTREME POINTS IN $H^1(U^n)$

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1. Introduction. Rudin and de Leeuw [1] characterized the extreme points of the unit ball in $H^1(U)$ as those outer functions in $H^1(U)$ whose norm is one. Rudin [5] has extended many properties of $H^1(U)$ to $H^1(U^n)$ and it is natural to ask for a characterization of extreme points of the unit ball of $H^1(U^n)$. As Yabuta [6] points out it is still true that outer functions are extreme, but he gives an example to show there are other extreme ponits with zeros in U^n. Riesenberg [3] has obtained characterizations for extreme points that are polynomials and we shall include his work here. The following lemma illustrates the approach we shall take to determine whether a function f is extreme.

LEMMA A. *f of norm 1 is not extreme in the unit ball of $H^1(U^n)$ iff there is an $h \in H^1(U^n)$ for which h/f is nonconstant, real and bounded* a.e. *on T^n.*

The proof of Lemma A is completely analogous to the proof in one variable. Lemma A is equivalent to the criterion that there exists an $h \in H^1(U^n)$ with arg $(h + f)$ = arg $(f - h)$ = arg (f) a.e. on T^n. We thus study the question of characterizing those $g \in H^1(U^n)$ for which arg (g) = arg (f) a.e. on T^n. In Section 2 we give such a characterization assuming that f is either analytic on $\overline{U^n}$ or continuous on $\overline{U^n}$ and nonzero on T^n. Our characterization reduces to a case covered by Yabuta [7] if f has no zeros on $\overline{U^n}$, but we are primarily interested in functions with zeros in U^n. In Section 3 we deduce Riesenberg's work from our characterizations. In Section 4 we obtain some new types of extreme points. We also examine the h that would exist from Lemma A if f were not extreme. We conclude that if $f \neq 0$ on T^n, then if f is in $A(U^n)$, then $h \in A(U^n)$, and if f is analytic on $\overline{U^n}$, then so is h.

2. Notation.

2.1. If Q is a polynomial in $\mathbf{C}^n$, $\tilde{Q}$ is the polynomial whose coefficients are the complex conjugates of the coefficients of Q. If $z = (z_1, \cdots, z_n)$ is in C^n with $z_i \neq 0$ for all i, then $1/z = (1/z_1, \cdots, 1/z_n)$.

DEFINITION 2.2 (Riesenberg). Q satisfies the symmetry condition with respect to a monomial $M(z)$ iff $M(z)\tilde{Q}(1/z) = Q(z)$.

It is not hard to derive (see Riesenberg [3]) that Q is symmetric with respect to some monomial M iff Q has the form

$$(1) \qquad Q(z) = \alpha z^{m(0)} \prod_{j=1}^{t} Q_j(z) z^{m(j)} \tilde{Q}_j\left(\frac{1}{z}\right) \prod_{j=t+1}^{l} Q_j(z).$$

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