LIE DERIVATIONS OF VON NEUMANN ALGEBRAS

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Let M be a von Neumann algebra and denote by $[M, M]$ the linear span of all operators of the form $[X, Y] = XY - YX$ for $X, Y \in M$. A mapping $L : [M, M] \to M$ is called a Lie derivation of $[M, M]$ if L is linear and $L[X, Y] = [L(X), Y] + [X, L(Y)]$ for $X, Y \in [M, M]$. It is shown that L can be extended to an associative derivation D of M, where $D(X) = [A, X]$, $A \in M$. As a corollary, if L is a Lie derivation of M, then $L(X) = [A, X] + \lambda(X)$, where $A \in M$ and λ is a linear map from M into Z_M which annihilates brackets of operators.

1. Introduction. Let M be an associative algebra over the complex field. With the multiplication $[X, Y] = XY - YX$, M can be considered a Lie algebra and its structure can be studied. A Lie subalgebra M_0 of M is a linear subspace of M closed under the bracket multiplication. A *Lie derivation* of M_0 into M is a linear map L which has the property that $L[X, Y] = [L(X), Y] + [X, L(Y)]$ for all X, Y in M_0. Martindale [3] has shown that if $M_0 = M$ is a primitive ring with nontrivial idempotent and characteristic not equal to 2, then L is of the form $D + \lambda$, where D is an associative derivation of M and λ is an additive map of M into its center which annihilates brackets of ring elements.

In this note we show that if M is a von Neumann algebra and if $M_0 = [M, M]$, the linear subspace of all finite linear combinations of elements of the form $[X, Y]$, $X, Y \in M$, then a Lie derivation L of $[M, M]$ in M can be extended to an associative derivation D of M. (It is known [7; Theorem 1] that if D is an associative derivation of a von Neumann algebra, then D is inner. That is, $D(X) = [A, X]$ for some $A \in M$.) As a corollary we have that a Lie derivation L of a von Neumann algebra M is of the form $D + \lambda$. Analogous results for extensions of Lie isomorphisms of $[M, M]$, where M is a simple ring, have been obtained in [2] and for extensions of Lie *-isomorphisms of $[M, M]$, where M is a von Neumann algebra, in [4].

We use Dixmier [1] as a reference for notation and general results concerning von Neumann algebras. If M is a von Neumann algebra, we denote by Z_M the center of M. If P and Q are projections in M, then $PMQ = \{PAQ \mid A \in M\}$, $PMQMP = \{\sum_{i=1}^{n} PX_iQY_iP \mid X_i, Y_i \in M\}$, and $M_P = PMP$.

2. Lie derivations of $[M, M]$. Let L: $[M, M] \to M$ be a Lie derivation of $[M, M]$, where M is a von Neumann algebra. Because of a "lack of room" in matrix computations we treat the I_2 case separately.

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