ON SET-VALUED METRIC PROJECTIONS, HAHN-BANACH EXTENSION MAPS, AND SPHERICAL IMAGE MAPS

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1. Introduction. Let E be a (real or complex) normed linear space and G a subset of E. The *metric projection* onto G is the mapping $P_G : E \to 2^G$ which associates with each $x \in E$ its set of nearest points in G. Thus

$$P_G(x) = \{g \in G \mid \|x - g\| = d(x, G)\},$$

where $d(x, G) = \inf \{\|x - g\| \mid g \in G\}$. G is called proximinal (respectively *Čebyšev*) if $P_G(x)$ contains at least (respectively exactly) one point for each $x \in E$.

In this paper we obtain some contributions to the theory of semicontinuity properties of metric projections. This study is along lines analogous to the studies made in [12] and [14]–[16]. In particular, we shall consider four different types of semicontinuity for set-valued mappings: upper and lower semicontinuity —concepts which go back at least to the 1920's (cf. Hahn [4] and the bibliography cited there)—and upper Hausdorff and lower Hausdorff semicontinuity, which were introduced and studied recently in [12].

In Section 2 we define these concepts and collect some basic facts about them. Included here are two embedding theorems, one of which yields an approximation theoretic characterization of finite dimensional spaces (Theorem 3). In Section 3 the semicontinuity of P_Γ, where Γ is a weak* closed linear subspace of E^*, is shown to be equivalent to the semicontinuity of its associated Hahn-Banach extension map (Theorem 5). In Section 4, which contains the main results of the paper, our study was motivated by one of the main theorems of Holmes [5; Theorem 14] which may be stated as follows.

THEOREM H. *Let E be a strictly convex reflexive Banach space and let G be a Čebyšev subspace of E having finite codimension. Then P_G is continuous if and only if $T|_{G^\perp}$ is continuous.*

(Here T denotes the spherical image mapping from E^* into 2^E defined by

$$T(f) = \{x \in E \mid f(x) = \|f\| \, \|x\|, \|x\| = \|f\|\}$$

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