ON THE RUDIN-SHAPIRO POLYNOMIALS

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1. Introduction. In an earlier paper [2] some of the basic properties of the Rudin-Shapiro (R-S) polynomials $P_n(x)$ and $Q_n(x)$ were developed. In the present paper we continue the investigation of these polynomials as well as the function $P(x)$ defined by the power series having $P_n(x)$ as its first 2^n terms.

The R-S polynomials are defined recursively by the formulas

$$P_{n+1}(x) = P_n(x) + x^{2^n}Q_n(x), \qquad n \geq 0, \tag{1}$$

$$Q_{n+1}(x) = P_n(x) - x^{2^n}Q_n(x), \qquad n \geq 0, \tag{2}$$

where $P_0(x) = Q_0(x) = 1$. (See [3] and [4].)

From [2] we have

$$Q_n(x) = (-1)^n x^{2^n-1} P_n(-1/x), \qquad n \geq 0, \tag{3}$$

$$P_{n+1}(x) = P_n(x^2) + xP_n(-x^2), \qquad n \geq 0, \tag{4}$$

$$Q_{n+1}(x) = Q_n(x^2) + xQ_n(-x^2), \qquad n \geq 1. \tag{5}$$

From [3] there is the relationship

$$|P_n(e^{i\theta})|^2 + |Q_n(e^{i\theta})|^2 = 2^{n+1}, \qquad n \geq 0, \quad \theta \text{ real}. \tag{6}$$

The investigation pursued in this paper is contained in eight sections. In Section 2 a pair of two-parameter identities involving $P_n(x)$ and $Q_n(x)$ will be proved. These will then be applied to evaluate $P_n(\pm 1)$ and $Q_n(\pm 1)$ in a simple way and to deduce an interesting reduction formula for $P_n(x)$ at the points $x = \exp(2\pi ir/2^m)$ on the unit circle.

In the third section elementary bounds for the complex roots of $P_n(x)$ and $Q_n(x)$ are given.

The next three sections are devoted to a study of the R-S polynomials on the real axis. In particular, it is established that they have exactly one real root. Formulas are also obtained for $P_n'(\pm 1)$, $Q_n'(\pm 1)$, $P_n''(\pm 1)$, and $Q_n''(\pm 1)$.

In the final three sections the function $P(x)$ is shown to have the unit circle as a natural boundary. Also, it is shown to have no roots on certain lines through the origin. Finally, it is demonstrated that the sequence of coefficients of the power series defining $P(x)$ possesses a simple orthogonality property.

2. Development of identities. (a) We begin by establishing a pair of two-parameter identities from which many of the results of this paper follow.