INHOMOGENEOUS CAUCHY-RIEMANN SYSTEMS WITH SMOOTH DEPENDENCE ON PARAMETERS

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1. Introduction. The purpose of this paper is to consider some of the classical existence problems of several complex variables when the given data depend smoothly on some parameters. We show that the solutions to these problems can be made to vary smoothly with the data. All the results considered here follow from a simple proposition on the solution of the inhomogeneous Cauchy–Riemann equations on a domain of holomorphy U when the right side consists of smooth functions on $U \times V$, V a differentiable manifold. These results can be used, in particular, to simplify the constructions given by Henkin [6] and Ramirez [7] of integral formulas for holomorphic functions on strongly pseudoconvex domains.

Before stating the main proposition we introduce the following notation. Let U and V be open sets in a Euclidean space. If $0 \leq j, k \leq \infty$, let $C^{j,k}(U \times V)$ denote the vector space of continuous, complex-valued functions $f(x, y)$ on $U \times V$ for which the mixed partial derivatives $D_x^a D_y^b f$ are continuous on $U \times V$ for all pairs (a, b) of multi-indices satisfying $|a| \leq j$ and $|b| \leq k$. With the topology of uniform convergence on compact subsets of $U \times V$ the space $C^{j,k}(U \times V)$ is a Fréchet space.

The main proposition is the following.

PROPOSITION 1. *Let U be a domain of holomorphy in C^n and let V be an open set in a Euclidean space. Let $f_1(z, w), \cdots, f_n(z, w)$ belong to $C^{\infty,k}(U \times V)$, $0 \leq k \leq \infty$, and suppose that the differential form $f = \sum f_i(z, w)\, d\bar{z}_i$ satisfies $\sum \bar{\partial}_z f_i(z, w) \wedge d\bar{z}_i = 0$ on $U \times V$. Then there exists $u \in C^{\infty,k}(U \times V)$ such that $\bar{\partial}_z u = f$.*

It will be clear from the proof that, with the obvious modifications, Proposition 1 remains valid when U is any Stein manifold, V is a sufficiently smooth σ-compact differentiable manifold (or, when $k = 0$, any σ-compact locally compact Hausdorff space), and f is a differential form of class $C^{\infty,k}$ which satisfies $\bar{\partial}_z f = 0$.

Proposition 1 and Propositions 2 and 3 below are related to theorems of Bishop [1] and Bungart [2] who studied holomorphic functions on a Stein manifold with values in a Fréchet space. Proposition 5 below is due to Henkin [6]. A somewhat more general result was proved by Ramirez [7; §4, Lemma 1] as a consequence of his division theorem.

Our results are not as general as those of Bishop, Bungart and Ramirez but