STRONG VARIATION FOR THE SAMPLE FUNCTIONS OF A STABLE PROCESS

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1. Introduction. Given a function $f : [0, s] \to R^d$ and a monotone increasing $\psi : R^+ \to R^+$ with $\psi(0) = 0$, we define the strong ψ-variation of f on $[0, t]$ by

$$V_\psi^*(f; 0, t) = \sup_\pi \sum_{t_i \in \pi} \psi(|f(t_i) - f(t_{i-1})|), \tag{1}$$

where the supremum is taken over all finite partitions $\pi = \{0 = t_0 < \cdots < t_m = t\}$ of $[0, t]$. Given a fixed f, one can ask, "For which functions ψ is $V_\psi^*(f; 0, t)$ finite?" In the present paper we are interested in the case where f is the sample path of a process with stationary independent increments (Lévy process) taking values in R^d. That is, we replace f by $X = X(\cdot\,, \omega)$ for a fixed ω in the underlying probability space, $X(s, \omega)$ being a Lévy process, and we ask, "For which functions ψ is $V_\psi^*(X; 0, t)$ finite with probability one?"

In [2] Blumenthal and Getoor define a numerical index $\beta \in [0, 2]$ for a Lévy process and show that if $\psi(s) = s^\gamma$, then $\gamma < \beta$ implies $V_\psi^*(X; 0, t) = \infty$ a.s. and $\beta < (\gamma \wedge 1)$ implies $V_\psi^*(X; 0, t) < \infty$ a.s. provided that the process has no drift. The gap in knowledge about the s^γ-variation of Lévy processes has only recently been filled by Monroe [12] who uses the method of subordination to Brownian motion and by Bretagnolle [3] who truncates the Lévy measure ν of the process (see Section 2 for definitions) and then shows that $E\{V_\psi^*(X; 0, t)\} < \infty$ when $\int_{-1}^{+1} \psi(|x|)\nu(dx) < \infty$ and ψ is sufficiently smooth and not too close to $\psi(s) = s^2$. Our approach and methods are different from those of Bretagnolle; we are able to deal with arbitrary variation functions ψ, but we require smoothness conditions on the Lévy measure ν. To simplify the writing we give the details for the class of stable processes of index α, $0 < \alpha < 2$, and then we indicate in Section 5 a wide class of processes for which the result is valid.

Instead of considering V_ψ^* directly we consider V_ψ^+, the limiting strong ψ-variation on $[0, 1]$ defined by

$$V_\psi^+(f; 0, t) = \lim_{\delta \to 0} \sup_{\sigma(\pi) < \delta} \sum_{t_i \in \pi} \psi(|f(t_i) - f(t_{i-1})|), \tag{2}$$

where $\sigma(\pi) = \max (t_i - t_{i-1})$, the mesh of the dissection π. Since our sample paths $X(t)$ are bounded almost surely, it is clear that $V_\psi^+(X; 0, t)$ is finite a.s. if and only if $V_\psi^*(X; 0, t)$ is finite a.s. One of us [14] showed that when $\psi(s) = s^2/2 \log \log s^{-1}$ and $X(s)$ is Brownian motion, then $V_\psi^+(X; 0, t) = t$ almost surely so that, in this case, there is a "correct" function ψ for measuring the

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