IDEAL C^*-ALGEBRAS

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The second dual A^{**} of the C^*-algebra A is itself a C^*-algebra. When does A^{**} contain A as an ideal? The major purpose of this paper is to present several answers to the question (Theorem 5.5) and to clarify the relations among the answers by a unified approach to their verification. The ideals of A are of primary interest and importance, so much so that a C^*-algebra which is an ideal in its second dual is, indeed, an "ideal" C^*-algebra. The paper brings together several results long known of C^*-algebras, several results contained in the author's 1968 dissertation [4], and several new results.

Section 1 reviews the pertinent facts about C^*-algebras and von Neumann algebras. Section 2 establishes and exploits several hereditary properties of ideal C^*-algebras. Section 3 treats ideal commutative C^*-algebras. Section 4 describes the ideals in an ideal C^*-algebra. Section 5 presents the characterizations we seek of ideal C^*-algebras. Section 6 suggests some extensions of our results.

1. Preliminaries. A *C^*-algebra* A is a Banach albegra with an involution $x \to x^*$ in which the norm of A and the involution are related by the vital property $||x^*x|| = ||x||^2$ for every x in A. Every commutative C^*-algebra is of the form $C_0(X)$, the space of all continuous, complex-valued functions vanishing at infinity on some locally compact, Hausdorff space X. Every C^*-algebra is isomorphic to a norm closed, self-adjoint sub-algebra of some $\mathfrak{L}(H)$, the algebra of all continuous linear operators on the Hilbert space H.

Denote by A^* the dual of the C^*-algebra A, that is, the Banach space of all continuous linear functionals on A; denote by A^{**} the second dual of A, the dual of A^*. Define the following linear functions:

$$(f, x) \to f \circ x : A^* \times A \to A^*, \quad \text{where} \quad (f \circ x)(y) = f(xy),$$

$$(F, f) \to F \circ f : A^{**} \times A^* \to A^*, \quad \text{where} \quad (F \circ f)(x) = F(f \circ x),$$

and

$$(F, G) \to F \circ G : A^{**} \times A^{**} \to A^{**}, \quad \text{where} \quad (F \circ G)(f) = F(G \circ f).$$

The multiplication thus defined on A^{**}, called *Arens' multiplication* after its inventor, extends the multiplication on A, is weak* continuous in A^{**} in each variable separately, and makes A^{**} a Banach algebra. Define

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