NON-OPENNESS OF LOCI IN NOETHERIAN RINGS

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1. Many loci which are Zariski open in the algebro-geometric case (nonsingular points, normal points, etc.) turn out not to be open for all Noetherian rings R. We give here a very general technique for constructing examples of rings possessing non-open loci. If $\mathcal{P}$ is a property, i.e., class, of Noetherian local rings, we refer to

$$\{P \in \operatorname{Spec} R : R_P \quad \text{has} \quad \mathcal{P}\}$$

as the $\mathcal{P}$ locus in Spec R, where R is a Noetherian ring. Our construction shows that very few interesting properties $\mathcal{P}$ lead to loci which are always open.

In [3; last sentence of 6.11] the question was raised, do there exist Noetherian rings of dimension greater than or equal to 2 which fail to satisfy (CMU). (A Noetherian ring R has property (CMU) if for every residue class domain D of R there is a nonzero $d \in D$ such that $D[1/d]$ is Cohen-Macaulay. It is shown in [3; 162–163, §6], where the terminology is "Spec R has (CMU)" rather than "R has (CMU)", that R has (CMU) if and only if for every R-module E of finite type

$$\{P \in \operatorname{Spec} R : E_P \quad \text{is Cohen-Macaulay over} \quad R_P\}$$

is open. Quotients and localizations of (CMU) rings have (CMU), and it is shown in [3] that if there is a Cohen-Macaulay module E over a Noetherian ring R such that Supp E = Spec R, i.e., $\operatorname{Ann}_R E$ is nilpotent, then R has (CMU). Hence, a homomorphic image of a Cohen-Macaulay ring has (CMU).) Ferrand and Raynaud [2] give an example of a local ring of dimension 3 which does not satisfy (CMU), thus answering Grothendieck's question.

We use our technique here to give two examples of Noetherian domains all of whose local rings are algebro-geometric and neither of which has (CMU). These rings fail to have (CMU) for reasons entirely different from those underlying the example of [2]; their local rings do have (CMU). We note that one has Krull dimension 2, and its integral closure is regular (but not module-finite). The other has Krull dimension 4 and is a UFD.

We also show that our technique gives a quite simple example of a one-dimensional locally algebro-geometric domain which has no maximal ideal in its normal (nonsingular) locus. Cf. [4].

2. Throughout the rest of this paper K denotes a field and all otherwise unspecified tensor products are taken over K. If R is a K-algebra, we say

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