INVARIANCE OF THE NONSTANDARD HULLS OF LOCALLY CONVEX SPACES

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In this paper we begin the systematic study of the class of locally convex topological vector spaces with invariant nonstandard hulls [2]. By definition this property, which arises in nonstandard analysis, is possessed by a locally convex space (E, θ) exactly when the nonstandard hulls $(\hat{E}, \hat{\theta})$ are all isomorphic to the completion of (E, θ). As was shown in [2], this is the only reasonable concept of invariance or stability for the nonstandard hulls of (E, θ). Indeed, if the nonstandard hulls of (E, θ) are not invariant in this sense, then for each cardinal number κ there is a nonstandard hull of (E, θ) which has cardinality greater than κ.

In [2] it was shown that if the nonstandard hulls of (E, θ) are invariant, then every θ-bounded subset of E is θ-totally bounded. The converse does not hold in general, as is shown by an example in Section 1. However, it does hold when (E, θ) is a metrizable space (Theorem 1). Therefore, among Fréchet spaces, the ones with invariant nonstandard hulls are exactly the Fréchet-Montel (FM) spaces. In particular, as was proved in [2], a Banach space has invariant nonstandard hulls if and only if it is finite dimensional. If θ is the strongest possible locally convex vector topology on E, then every θ-bounded set is relatively θ-compact. However, the nonstandard hulls of such a space are invariant if and only if the cardinality of a Hamel basis for E is smaller than the first measurable cardinal (Theorem 2).

If (E, θ) is a Schwarz space, in particular if it is a nuclear space, then the nonstandard hulls of (E, θ) are invariant (Theorem 4). Also, in Section 3 a sufficient condition is given, involving the possibility of defining θ in terms of locally (weakly) convergent sequences in the dual space $(E, \theta)'$, which implies that (E, θ) has invariant nonstandard hulls. This result implies that if E is the dual space of a Fréchet-Montel space F and if θ is the strong topology $\beta(E, F)$, then (E, θ) has invariant nonstandard hulls.

As an application of this property it is shown that certain Banach spaces are dual spaces of Banach spaces with dense subsets of known cardinality. The argument uses the fact that these Banach spaces are naturally associated with locally convex spaces which have invariant nonstandard hulls. This result includes the result of Rubel and Shields [8] that the Banach space $H^\infty(D)$ of bounded analytic functions (on the complex domain D) is the dual space of a separable Banach space. Our proof of this fact hinges on the observation that if θ is the topology of uniform convergence on compact subsets of D, then $(H^\infty(D), \theta)$ has invariant nonstandard hulls and is metrizable.

Received October 24, 1972.