ON GAUSSIAN SUMS AND OTHER EXPONENTIAL SUMS WITH PERIODIC COEFFICIENTS

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1. Introduction. L. Kronecker apparently was the first person to apply contour integration to the study of Gaussian sums; in [6] he evaluated the ordinary Gaussian sum

$$\sum_{n=0}^{q-1} e^{2\pi i n^2/q} \tag{1.1}$$

by contour integration. L. J. Mordell [7] has also evaluated (1.1) by employing contour integration. (Mordell published his proof again in [8; 351–352].) This elegant proof of Mordell deserves wider recognition, for it is considerably simpler than Kronecker's.

In this paper we consider exponential sums with periodic coefficients. Primary attention will be given to the special cases of Gaussian sums and character sums. All of our proofs use contour integration.

Let $A = \{a_n\}$, $-\infty < n < \infty$, be a sequence of complex numbers with period k, i.e., $a_n = a_{n+k}$ for every integer n. Since A has period k, a_n may be expanded in a finite Fourier series

$$a_n = \sum_{j=0}^{k-1} b_j e^{2\pi i j n/k}. \tag{1.2}$$

In fact, an elementary calculation shows that (1.2) is true if and only if

$$b_n = \frac{1}{k} \sum_{j=0}^{k-1} a_j e^{-2\pi i j n/k}, \qquad 0 \le n \le k-1. \tag{1.3}$$

We extend the definition of b_n to $-\infty < n < \infty$ by (1.3), i.e., by periodicity, and we put $B = \{b_n\}$. Furthermore, let

$$G(z, B) = \sum_{j=0}^{k-1} b_j e^{2\pi i j z/k}.$$

Observe that if $z = n$ is an integer,

$$G(n, B) = a_n . \tag{1.4}$$

If $\chi(n)$ denotes a character of modulus k, let

$$G(z, \chi) = \sum_{j=1}^{k-1} \chi(j) e^{2\pi i j z/k}$$

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