THE MACKEY-BLATTNER THEOREM AND TAKESAKI'S GENERALIZED COMMUTATION RELATION FOR LOCALLY COMPACT GROUPS

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1. Introduction. Let G be a locally compact group with left-invariant Haar measure μ, and let H and K be two closed subgroups of G. Let U and V denote the left and right regular representations of G on $L^2(G, \mu)$ respectively, and let π denote the usual representation of $L^\infty(G, \mu)$ on $L^2(G, \mu)$. Define $M(G/H, U(K))$ (respectively $M(K\backslash G, V(H))$) to be the von Neumann algebra on $L^2(G, \mu)$ generated by the operators $\pi(\varphi)$, $\varphi \in C_0(G/H)$, and the operators $U(r)$, $r \in K$ (respectively $\pi(\varphi)$, $\varphi \in C_0(K\backslash G)$, and $V(p)$, $p \in H$). Here an element of $C_0(G/H)$ (respectively $C_0(K\backslash G)$), the continuous complex-valued functions with compact support on the homogeneous space G/H of left H-cosets (respectively $K\backslash G$ of right K-cosets), is regarded as a function on G in the obvious manner. M. Takesaki has shown that

$$M(G/H, U(K))' = M(K\backslash G, V(H)) \tag{1}$$

whenever G is separable [10]. The following two well-known commutation relations are both special cases of (1):

$$U(G)' = V(G)'' \quad \text{and} \quad \pi(L^\infty(G, \mu))' = \pi(L^\infty(G, \mu)). \tag{2}$$

It turns out that (1) is closely related to the imprimitivity theorem. If L is a continuous unitary representation of H and if U^L is the representation of G obtained by inducing L up to G, then $L(H)'$ is *-isomorphic to a certain subalgebra of $U^L(G)'$ [2], [8]. For the remainder of this discussion this part of the imprimitivity theorem will be referred to as the *Mackey-Blattner theorem.* For an arbitrary G one can obtain the special case of (1) in which $K = G$ from the Mackey-Blattner theorem, and conversely, Takesaki's proof of (1) actually contains a proof of a generalized Mackey-Blattner theorem for the special case in which G is separable and L is the left regular representation of H.

There are several reasons why it would be desirable to obtain a proof of (1) in the general case. Firstly, an understanding of (1) will lead to a better understanding of (2), which has in turn played a crucial role in Takesaki's "converse" of Tatsuuma's duality theorem [9] (see also [11]). Secondly, it is possible to base a simple proof of [12; Theorem 6] on (1), and consequently one may hope that (1) will prove useful in subsequent studies of the role played by subgroups in the duality theory of locally compact groups.

The present paper contains two results. The first is a formulation and proof

Received May 25, 1972.