ON A CLASS OF ARITHMETICAL FUNCTIONS

JEAN-MARIE DE KONINCK

1. Introduction. Let $\omega(n)$ be the number of distinct prime divisors of n. Then estimates for $\sum_{n\leq x} \omega(n)$ are well known [3]. On the other hand, estimates for $\sum'_{n\leq x} 1/\omega(n)$ were only recently studied [1], [2]. (From here on, the prime in a sum of the form $\sum'_{n\leq x} 1/f(n)$ means that the sum is taken over all $n \leq x$ such that $f(n) \neq 0$.)

Using Turan's inequality, R. L. Duncan proves in [1] that

$$\sum_{n\leq x}' \frac{1}{\omega(n)} = O\left(\frac{x}{\log \log x}\right)$$

and then uses this result to show that $\Omega(n)/\omega(n)$ has average order one, where $\Omega(n)$ stands for the total number of prime divisors of n.

In this paper, we obtain a much better estimate for $\sum'_{n\leq x} 1/\omega(n)$ and we also obtain estimates for $\sum'_{n\leq x} 1/(f(n))^k$ for a large class of arithmetical functions $\{f(n)\}$ and an arbitrary positive integer k.

2. A result of A. Selberg and basic definitions. Before defining our class of functions, we state a result of A. Selberg [4]. Restricted to the particular case needed here the result may be stated as follows.

THEOREM A (Selberg). *Let* $g(s, t) = \sum_{n=1}^{\infty} b_t(n)/n^s$ *for* Re $s = \sigma > 1$, *and let* $\sum_{n=1}^{\infty} |b_t(n)|\, n^{-1} \log^{B+3} 2n$ *be uniformly bounded for* $|t| \leq B$. *Next, set* $(\zeta(s))^t g(s, t) = \sum_{n=1}^{\infty} a_t(n)/n^s$ *for* $\sigma > 1$. *Then we have* $\sum_{n\leq x} a_t(n) = (g(1, t)/\Gamma(t))\, x \log^{t-1}x + O\,(x \log^{t-2}x)$ *uniformly for* $|t| \leq B$, $x \geq 2$. *(Here* $\zeta(s)$ *stands for the Riemann zeta function.)*

DEFINITION 1. Let S denote the set of all real-valued arithmetical functions satisfying the following two conditions.

(1) $f(n) \neq 0 \Rightarrow f(n) \geq 1$ for each integer $n \geq 1$.

(2) $\displaystyle\sum_{\substack{n\leq x\\ f(n)=0}} 1 = O\left(\frac{x}{\log x}\right)$.

DEFINITION 2. Given α (from now on, unless otherwise mentioned, α stands for an arbitrary positive integer), we denote by S_α the set of those functions in S for which $t^{f(n)} = a_t(n)$ satisfies the conditions of Theorem A, with $B = 1$ and $D(t) = (g(1, t)/\Gamma(t)) \in C^{\alpha+1}[0, 1]$.

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