SOME SIMULTANEOUS EQUATIONS IN MATRICES

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Let A_1, A_2, A_3 be square matrices of dimension $r_1 \times r_1$, $r_2 \times r_2$, $r_3 \times r_3$ respectively. Necessary and sufficient conditions for the existence of X_1, X_2, X_3, satisfying $A_1 = X_1X_2X_3$, $A_2 = X_2X_3X_1$, $A_3 = X_3X_1X_2$, are (the Flanders conditions) that the elementary divisors of A_i corresponding to nonzero proper values be the same and that the elementary divisors corresponding to the proper value 0 deviate in degree by at most one unit. For $r > 3$ conditions for the solvability of $A_i = (\prod_{j=i}^{r} X_j)(\prod_{1}^{i-1} X_j)$ are more complicated; an extra condition on the degrees of the elementary divisors corresponding to 0 is involved.

1. Introduction. This article is a continuation of [2] and is a complement to Flanders' article [3]. See also [1], [4], [5]. The coordinate-free background is the following. Let U_i be r finite-dimensional vector spaces over the complex numbers, $i = 1(1)r$; let A_i be a given linear transformation of U_i into itself. Do there exist r linear transformations X_i, where X_i maps U_i into U_{i+1} (and $U_{r+1} \equiv U_1$), such that all the relations

$$
\begin{aligned}
A_1 &= X_1X_2 \cdots X_r \\
A_2 &= X_2 \cdots X_rX_1 \\
&\vdots \\
A_i &= X_i \cdots X_rX_1 \cdots X_{i-1} \\
&\vdots \\
A_r &= X_rX_1 \cdots X_{r-1}
\end{aligned}
\tag{1.01}
$$

hold? If U_i are all identical and A_i are all invertible (equivalent to nonsingular), Equations (1.01) are solvable if and only if A_i are similar [2]. For $r = 2$ the problem was completely solved by H. Flanders. Necessary and sufficient conditions for solvability of $A_1 = X_1X_2$, $A_2 = X_2X_1$ are (i) A_1, A_2 have the same nonzero proper values; (ii) the elementary divisors of A_1, A_2 corresponding to every nonzero proper value are the same; (iii) if the degrees of the elementary divisors corresponding to the proper value 0 for A_1 are $m_1 \geq m_2 \geq \cdots$ and for A_2 are $n_1 \geq n_2 \geq \cdots$, then for every ν, $|m_\nu - n_\nu| \leq 1$; and if for some ν there is no n_ν [m_ν], then m_ν [n_ν] is 1. Permissible sets are for example (3, 2, 1, 1) for the m_ν, (3, 3) for the n_ν. In this article, conditions (i), (ii), (iii) are called the Flanders conditions.

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