PARTITIONS AND RAMANUJAN'S CONTINUED FRACTION

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Let

$$x_n = 1 - x^n, \qquad x_n! = x_1 x_2 \cdots x_n .$$

The main result of this note is that

$$1 + \frac{ax}{1+}\frac{ax^2}{1+}\cdots\frac{ax^n}{1} = \frac{P_n(a, x)}{Q_n(a, x)},$$

where

$$(1) \qquad \begin{aligned} P_n(a, x) &= \sum_{r=0}^{[(n+1)/2]} a^r x^{r^2} \frac{x_{n-r+1}!}{x_r!\, x_{n-2r+1}!} \\ Q_n(a, x) &= \sum_{r=0}^{[n/2]} a^r x^{r(r+1)} \frac{x_{n-r}!}{x_r!\, x_{n-2r}!}. \end{aligned}$$

Letting $n \to \infty$, we obtain, if $|x| < 1$,

$$1 + \frac{ax}{1+}\frac{ax^2}{1+}\cdots = \sum_{r=0}^{\infty} a^r \frac{x^{r^2}}{x_r!} \Big/ \sum_{r=0}^{\infty} a^r \frac{x^{r(r+1)}}{x_r!},$$

a well-known result [1; 19.15.1].

We shall require the following result on partitions [2; Art. 241]

$$(2) \qquad \sum_k p(k, r, n) x^k = \frac{x_{n+r}!}{x_r!\, x_n!},$$

where $p(k, r, n)$ is the number of partitions of k into at most r parts not exceeding n.

It is standard that we can write

$$1 + \frac{\epsilon_1}{1+}\frac{\epsilon_2}{1+}\cdots\frac{\epsilon_n}{1} = \frac{P_n}{Q_n},$$

where

$$P_0 = 1, \quad Q_0 = 1, \quad P_1 = 1 + \epsilon_1, \quad Q_1 = 1,$$

and

$$P_r = P_{r-1} + \epsilon_r P_{r-2}, \qquad Q_r = Q_{r-1} + \epsilon_r Q_{r-2} \quad \text{for} \quad r \geq 2.$$

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