REGULARITY AND DISPERSION IN COUNTABLE SPACES

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The main aim of this paper is to answer a question raised by Prabir Roy [11] concerning the points of regularity of a countable space with a dispersion point. As a preliminary result, we also characterize a class of spaces admitting dispersion points (Theorem 3).

The concept of a dispersion point was first introduced by Knaster and Kuratowski [8]. A point x of a connected Hausdorff space X is called a dispersion point if $X \backslash \{x\}$ is totally disconnected. Since a countable connected Hausdorff space cannot be regular, we rarely come across countable connected Hausdorff spaces; still rarer are those with dispersion points. The first example of a countable Hausdorff space with a dispersion point was given by Martin [9]. Further examples have later been given by Gustin [2] and Miller [10]. It is natural to ask how strong a separation axiom can hold in these spaces. Answering a question of Martin [9] in this direction, Roy [11] gave the first example of a countable Urysohn space with dispersion point. A modification of this example has been used by Jones and Stone [3]. A simpler example was later given in [5] and improved to a better result in [7]. In all these examples there is at most one point of regularity. It can be proved that the points of regularity in such spaces must have empty interior. Roy [11] asked whether such spaces can be regular at a dense set of points. Theorem 4 below gives an affirmative answer. Theorem 10 shows that such spaces exist in plenty.

DEFINITION 1. A totally disconnected Hausdorff space X is said to admit a regular dispersion point if there exists a space Y with dispersion point y such that

(a) Y is regular at y and Y is Hausdorff

and (b) X is homeomorphic to $Y \backslash \{y\}$.

Remark 2. It can be easily seen that not every totally disconnected Hausdorff space can admit a regular dispersion point. For example, a zero-dimensional space cannot. A clopen set is a set that is both open and closed.

THEOREM 3. *A countable totally disconnected Hausdorff space X admits a regular dispersion point if and only if X has a countable open cover $\{\mathcal{U}_n \mid n = 1, 2, \cdots\}$ such that*

(1) $\overline{\mathcal{U}_n} \subset \mathcal{U}_{n+1}$ *for every* $n = 1, 2, \cdots$

and (2) *for every n it is true that no nonvoid subset of $\mathcal{U}_n$ is clopen in X.*

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