GOING DOWN

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Introduction. A well-known theorem of Krull [2] states that if the domain T is an integral extension of the integrally closed domain R, then going down holds in $R \subset T$. This paper investigates when going down holds in $R \subset T$, where R is a commutative domain with identity and T is a domain between R and its integral closure. If R is Noetherian, it is shown that $R \subset T$ has going down exactly when each prime of R of rank greater than one has a unique prime of T lying over it. An example is given which shows that the Noetherian assumption is needed. If R is not Noetherian, we can still make progress by assuming that $R[x] \subset T[x]$ has going down, where x is an indeterminate. It is proved that going down between those polynomial rings occurs exactly when each prime of $R[x]$ has a unique prime of $T[x]$ lying over it. The last result strengthens work done in [3] in which two indeterminates were used.

Definitions and Preliminaries. Let $R \subset T$ be domains. We will say this extension has *going up*, *lying over* or *going down* in accordance with the definitions of [1; Sections 1–6]. In particular, if P is prime in R, Q is prime in T, and $Q \cap R = P$, we say Q *lies over* P. If there is a unique prime of T lying over P in R, we say P is *unibranched* in T. If each prime of R is unibranched in T, we say $R \subset T$ is *unibranched*. Finally if u is in the quotient field of R, we call $\{r \in R \mid ru \in R\}$ the conductor of u in R. Note that since $u = a/b$ with a and b in R, $b \neq 0$, the conductor is nonzero and contains b.

The following two lemmas are straightforward and are stated without proof.

Lemma 1. (1) *If $R \subset S \subset T$ and going down holds in $R \subset S$ and $S \subset T$, then it holds in $R \subset T$.* (2) *If going down holds in $R \subset T$ and lying over holds in $S \subset T$, then going down holds in $R \subset S$.*

Lemma 2. *Let R be a domain and let u be in the quotient field of R. Suppose I is the conductor of u in R. If $I \not\subset P$, where P is a prime of R, then there is exactly one prime of $R[u]$ lying over P.*

Theorem 1. *Let R be a domain and let R' be its integral closure. Then going down holds in $R \subset R'$ if and only if going down holds in $R \subset T$ for any domain T which is an integral extension of R.*

Proof. Suppose $R \subset R'$ has going down. Observe that $R' \subset R'T$ is an integral extension. By [2] going down holds in $R' \subset R'T$. Lemma 1 implies $R \subset R'T$ has going down and then tells us $R \subset T$ has going down since $T \subset R'T$, being integral, has lying over.

The converse is immediate.

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