PRODUCT SPACES WITH COMPACTNESS-LIKE PROPERTIES

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1. Introduction. The well-known theorem of Tychonoff states that the product of any number of compact spaces is compact. On the other hand, many properties weaker than but similar to compactness (in particular, countable compactness and the Lindelöf property) are not even preserved by finite products. It is an interesting problem, therefore, to find conditions under which products will be countably compact or Lindelöf. This is just an example of a host of similar problems which arise from considering, in addition to countable compactness and Lindelöf, other compactness-like properties such as initial $\mathfrak{m}$-compactness, final $\mathfrak{m}$-compactness, and in general $[\mathfrak{m}, \mathfrak{n}]$-compactness (these terms are defined below). In this paper we give two sufficient conditions for products to have the compactness-like properties mentioned above. These conditions were motivated in part by the hypotheses of the following three theorems, and we shall show below that these theorems may be considered as special cases of a single result, namely, Theorem 1.4.

THEOREM 1.1 (C. T. Scarborough and A. H. Stone [11; 144, Theorem 5.5]) *Every product of at most $\aleph_1$ sequentially compact spaces is countably compact.*

This result has been extended to the following theorem.

THEOREM 1.2. (V. Saks and R. M. Stephenson, Jr. [10; 281, Theorem 2.4]) *Every product of at most $\aleph_1$ spaces in which every sequence has a subsequence with compact closure is countably compact.*

The definitions needed for the next theorem are stated below.

THEOREM 1.3. (N. Noble [9; 179, Theorem 4.4]) *Every product of at most $\mathfrak{m}^+$ initially $\mathfrak{m}$-compact spaces of character less than or equal to $\mathfrak{m}$ is initially $\mathfrak{m}$-compact.*

DEFINITIONS AND NOTATION. Throughout this paper the letters $\mathfrak{m}$ and $\mathfrak{n}$ will stand for infinite cardinal numbers with $\mathfrak{m} \leq \mathfrak{n}$, and $\mathfrak{m}^+$ will denote the first cardinal number strictly larger than $\mathfrak{m}$. For any set S let $|S|$ stand for the cardinal of S. If $\mathfrak{F}$ and $\mathfrak{G}$ are two collections of subsets of a set X, then $\mathfrak{F} \vee \mathfrak{G}$ stands for the set of all finite intersections of members of $\mathfrak{F}$ and $\mathfrak{G}$. A filter base $\mathfrak{G}$ is said to be *finer* than a filter base $\mathfrak{F}$ provided every member of $\mathfrak{F}$ contains a member of $\mathfrak{G}$. A space has *character* less than or equal to $\mathfrak{m}$ provided that every point in the space has a fundamental system of neighborhoods of cardinality less than or equal to $\mathfrak{m}$. A space is said to be *initially $\mathfrak{m}$-compact*

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