UP-DOWN SEQUENCES

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1. Introduction. We recall that an up-down permutation of $Z_n = \{1, 2, \cdots, n\}$ is a permutation $(a_1, a_2, \cdots, a_n)$ such that

$$a_1 < a_2 , a_2 > a_3 , a_3 < a_4 , a_4 > a_5 , \cdots .$$

Similarly a down-up permutation is one in which

$$a_1 > a_2 , a_2 < a_3 , a_3 > a_4 , a_4 < a_5 , \cdots .$$

If $(a_1, a_2, \cdots, a_n)$ is an up-down permutation and

$$b_i = n - a_i + 1, \qquad i = 1, 2, \cdots, n,$$

then $(b_1, b_2, \cdots, b_n)$ is a down-up permutation. Hence, for $n > 1$ the number of up-down permutations is equal to the number of down-up permutations.

Let $A(n)$ denote the number of up-down permutations of Z_n. It is known [3; 105–112] that

$$\sum_{n=0}^{\infty} A(2n) \frac{x^{2n}}{(2n)!} = \sec x, \sum_{n=0}^{\infty} A(2n+1) \frac{x^{2n+1}}{(2n+1)!} = \tan x, \tag{1.1}$$

$$A(0) = A(1) = 1.$$

The present paper is concerned with the following problem. Let $s_1, s_2, \cdots, s_n$ denote nonnegative integers and put

$$N = s_1 + s_2 + \cdots + s_n . \tag{1.2}$$

We consider sequences

$$\sigma = (a_1, a_2, \cdots, a_N) \tag{1.3}$$

of length N, where $a_i \in Z_n$ and the element j occurs exactly s_j times. We call $[s_1, s_2, \cdots, s_n]$ the *specification* of σ. We shall say that σ is an up-down sequence provided

$$a_1 < a_2 , a_2 > a_3 , a_3 < a_4 , a_4 > a_5 , \cdots \tag{1.4}$$

and similarly for down-up sequences.

Corresponding to the up-down sequence σ defined by (1.3) we have the down-up sequence

$$\sigma' = (b_1, b_2, \cdots, b_N), \tag{1.5}$$

where

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