ON THE NUMBER OF DISTINGUISHED REPRESENTATIONS OF A GROUP ELEMENT

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1. Introduction. Let Δ denote a property which an element of a group may possess. We are interested in the number of representations of an element in a finite group as a product of r elements possessing Δ.

More generally, let D be a nonempty subset of a finite group G and denote by $N_r^D(a) = N_r(a)$ the number of solutions of the equation

$$x_1 \cdots x_r = a \tag{1}$$

where a is an element of G and $x_1, \cdots, x_r$ belong to D. Of course, if a is not in the subgroup generated by D, then $N_r(a) = 0$ for all r.

In Proposition 1 we note that the evaluation of $N_r^D(a)$ reduces to the corresponding question for a certain quotient group of G.

If D itself is a subgroup of G, then trivially $N_r(a) = |D|^{r-1}$ for a in D.

For arbitrary D the calculation of $N_r^D(a)$ seems quite difficult. One of our main results is the explicit determination in Theorem 2 of $N_r^D(a)$ when $G \setminus D$, the complement of D in G, is a subgroup of G.

As an application of Theorem 2 we obtain in Corollary 5 the number of representations of a given element in an abelian group G as a product of r elements of maximal order in G. In particular, for G cyclic this number agrees with a formula derived by Rearick [3] and is essentially equivalent to an earlier formula of Dixon [1].

In §4 analogous questions for rings are considered.

2. Main results. For D a nonempty subset of G let $J(D) = J$ denote the largest normal subgroup of G such that $xJ \subseteq D$ for all x in D. We say that G is *D-reduced* if $J = \{1\}$. If $a \varepsilon G$, let $\bar{a}$ denote aJ in $\bar{G} = G/J$.

PROPOSITION 1. *If $\bar{D} = \{\bar{x} \mid x \varepsilon D\}$, then $\bar{G}$ is $\bar{D}$-reduced and*

$$N_r^D(a) = |J|^{r-1} N_r^{\bar{D}}(\bar{a}). \tag{2}$$

Proof. If $\bar{K} = K/J$ is the largest normal subgroup of $\bar{G}$ such that $\bar{x}\bar{K} \subseteq \bar{D}$ for all $\bar{x}$ in $\bar{D}$, then clearly $xK \subseteq D$ for all x in D. However, K is a normal subgroup of G and thus $K = J$, which establishes that $\bar{G}$ is $\bar{D}$-reduced.

To prove (2) we require the following lemma.

LEMMA. *Let G be a finite group and J a normal subgroup of G. If $x_1, \cdots, x_r$ belong to G, then the number of r-tuples $(y_1, \cdots, y_r)$ satisfying $x_1 \cdots x_r = y_1 \cdots y_r$ and $y_i \varepsilon x_i J$ for $i = 1, \cdots, r$ is equal to $|J|^{r-1}$.*

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