ON McSHANE'S VECTOR-VALUED INTEGRAL

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E. J. McShane has recently given an abstract version [3] of a generalization of the classical Riemann integral due (independently) to J. Kurzweil and R. Henstock. Simultaneously, Henstock's own abstract version [2] has appeared. Although there is much in common between these two treatments there are two distinct ideas which do not overlap in their presentations, namely, Henstock's notion of "variation" and McShane's concept of "absolute integrability".

In this paper we show how McShane's "absolute integrability" can be studied in a context derived mainly from Henstock's version of the abstract theory. In particular the underlying measure theory is shown to emerge quite naturally from this point of view.

Notation. The notation used is similar to that of the preceding paper except that the system $(T, \mathfrak{A}, \mathbf{I})$ will arise out of some algebraic structures and the resulting theory will then have more algebraic properties. Let $\mathbf{B}$ be a nonempty class of subsets of a set T. We require the following.

$\mathbf{B}$ is called a *clan* if $A \setminus B \in \mathbf{B}$ and $A \cup B \in \mathbf{B}$ for all $A, B \in \mathbf{B}$.

$\mathbf{B}$ is called a *semiclan* if (i) $A \cap B \in \mathbf{B}$ for $A, B \in \mathbf{B}$ and (ii) for every pair (A, B) of sets of $\mathbf{B}$ for which $A \subseteq B$ there exists a finite family $\{C_0, C_1, C_2, \cdots, C_n\}$ of sets of $\mathbf{B}$ such that $A = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_n = B$ and $C_i \setminus C_{i-1} \in \mathbf{B}$ for $i = 1, 2, \cdots, n$.

$\mathbf{B}$ is called a *tribe* if (i) $A \setminus B \in \mathbf{B}$ for $A, B \in \mathbf{B}$ and (ii) $\bigcup_{i=1}^{\infty} A_i \in \mathbf{B}$ for every sequence $\{A_i\}$ of sets in $\mathbf{B}$.

$\mathbf{B}$ is called a *semitribe* if (i) $A \setminus B \in \mathbf{B}$ for $A, B \in \mathbf{B}$, (ii) $A \cup B \in \mathbf{B}$ for $A, B \in \mathbf{B}$ and (iii) $\bigcap_{i=1}^{\infty} A_i \in \mathbf{B}$ for every sequence $\{A_i\}$ of sets in $\mathbf{B}$.

1. Partitioning systems. Let T be a set, $\mathfrak{B}$ a semiclan [1; 7] (semiring) of subsets of T and $\mathfrak{E}$ the clan (ring) generated by $\mathfrak{B}$. The sets in $\mathfrak{E}$ will be called elementary sets. We write $\mathbf{I} = \mathfrak{B} \times T$. A finite subset $\mathbf{D}$ of $\mathbf{I}$ is called a *partition* if the sets $\{I : (I, x) \in \mathbf{D}\}$ are disjointed. We write then $\sigma(\mathbf{D}) = \bigcup \{I : (I, x) \in \mathbf{D}\}$ and we call $\mathbf{D}$ a *partition of the elementary set* $\sigma(\mathbf{D})$. A subset $\mathbf{S}$ of $\mathbf{I}$ is said to partition E if $\mathbf{S}$ contains a partition of E.

For any subset $\mathbf{S}$ of $\mathbf{I}$ and for any family $\mathfrak{A}$ of subsets of $\mathbf{I}$ we denote

$$\mathbf{S}[X] = \{(I, x) \in \mathbf{S} : x \in X\}$$
$$\mathbf{S}(X) = \{(I, x) \in \mathbf{S} : I \subseteq X\}$$
$$\mathfrak{A}[X] = \{\mathbf{S}[X] : \mathbf{S} \in \mathfrak{A}\}$$
$$\mathfrak{A}(X) = \{\mathbf{S}(X) : \mathbf{S} \in \mathfrak{A}\}.$$

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