APPLICATIONS OF WEAK* SEMICONTINUITY IN C*-ALGEBRA THEORY

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In this paper we generalize the concept of a universally measurable function on a locally compact space to an arbitrary C^*-algebra. We show that the universally measurable operators associated with a C^*-algebra A, regarded as a subset of the second dual, can be isometrically represented in the atomic representation of A. Since the set of universally measurable operators contains the Borel operators, this allows us to study the Borel structure on the spectrum $\hat{A}$ of A induced by the central Borel operators. We show that each representation π of A gives rise to a central projection-valued spectral measure μ on $\hat{A}$ with this new Borel structure, and we give conditions which ensure that each element in the center of $\pi(A'')$ is of the form $\int f\, d\mu$ for some measurable function f on $\hat{A}$.

1. Introduction. In the paper [13] S. Kaplan has shown how various classes of functions on a locally compact space X, such as the Baire functions, the Borel functions, and the universally measurable functions, have simple characterizations when regarded as subsets of the second dual of $C_0(X)$. Using a construction of R. V. Kadison from [10] the present author defined in [15] the Baire operators associated with any (non-commutative) C^*-algebra A as the monotone sequential closure of A in A'', and F. Combes further extended Kaplan's ideas by introducing in [2] the weak* lower semicontinuous operators associated with A as the class A^m of elements in A'' which can be approximated weakly from below with self-adjoint operators of the form $x + \alpha$ with x in A and α in $\mathfrak{R}$. This then produced the class of Borel operators associated with A as the monotone sequential closure of $A^m + A_m$ in A'', where $A_m = -A^m$.

In this paper we take up the principle from [2] of regarding A^m as the analogue of the lower semicontinuous (l.s.c.) functions. We prove in §2 some auxilliary facts about A^m and show that when $1 \notin A$, then the set $A^m \cap A_m$ consists exactly of the self-adjoint, two-sided multipliers of A in A''. If A is simple, then $A^m \cap A_m$ is identified with the self-adjoint part of the derived algebra of A introduced in [21].

It is shown in §3 that we can extend the notion of universally measurable functions to the non-commutative situation by defining a self-adjoint element in A'' to be universally measurable if it can be approximated weakly from above with l.s.c. elements and from below with u.s.c. elements. The class of universally measurable operators turns out to be a norm-closed vector space, closed under strong sequential limits, and it is isometrically represented in the atomic representation of A.

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