TANGENT NUMBERS AND OPERATORS

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1. Introduction and summary. Put

$$\tan x = \sum_{n=0}^{\infty} T_n x^n/n!, \qquad T_{2n} = 0. \tag{1.1}$$

It is well known that $T_{2n+1} > 0$ but this is by no means obvious from (1.1). A simple way of proving this result is the following. Differentiation of

$$\tan(\arctan x) = x \tag{1.2}$$

gives

$$\sum_{n=0}^{\infty} T_{n+1}(\arctan x)^n/n! = 1 + x^2.$$

A second differentiation gives

$$\sum_{n=0}^{\infty} T_{n+2}(\arctan x)^n/n! = 2x(1 + x^2)$$

while a third yields

$$\sum_{n=0}^{\infty} T_{n+3}(\arctan x)^n/n! = (2 + 6x^2)(1 + x^2)$$

and so on. After k steps we get the formula

$$T_k = ((1 + x^2)D)^n\, x|_{x=0} \tag{1.3}$$

which evidently yields the desired result.

In view of (1.3) it is natural to consider the expansion of the operator $((1 + x^2)D)^n$. We put

$$((1 + x^2)D)^n = \sum_{k=0}^{n} P_{n,k}(x)(1 + x^2)^k D^k, \tag{1.4}$$

where the $P_{n,k}(x)$ are polynomials in x. We show that

$$P_{n,k}(x) = \frac{1}{k!\,(k-1)!} D^{k-1}P_{n,1}(x), \qquad 1 \leq k \leq n, \tag{1.5}$$

and

$$\sum_{n=1}^{\infty} P_n(x)z^n/n! = \frac{\tan z}{1 - x\tan z}, \qquad P_n(x) \equiv P_{n,1}(x). \tag{1.6}$$

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