BASIC INTERVALS IN THE LATTICE OF TOPOLOGIES

RICHARD VALENT AND ROLAND E. LARSON

Let Σ be the set of all topologies on a set X and let $\leq$ be the partial ordering of inclusion. Under this ordering Σ is a complete lattice. Given $\mathfrak{J}_1, \mathfrak{J}_2 \in \Sigma$, $\mathfrak{J}_1 \wedge \mathfrak{J}_2$ is the set intersection of $\mathfrak{J}_1$ with $\mathfrak{J}_2$ and $\mathfrak{J}_1 \vee \mathfrak{J}_2$ is the smallest topology on X containing $\mathfrak{J}_1 \cup \mathfrak{J}_2$. When Birkhoff introduced Σ in 1936 he noted that the lattice of T_1-topologies on X, which we call Λ, is an interval in Σ [1].

In this paper we will introduce and investigate a class of intervals of Σ which we call the basic intervals of Σ. Although these intervals arise quite naturally throughout the entire complex structure of Σ, they have the pleasant lattice properties of being Brouwerian, infinitely meet distributive, compactly generated, and complete. To begin the first of the four sections of this paper we will topologically introduce these intervals and then present a lattice characterization which proves to be fruitful. To conclude the first section we prove that a basic interval in Σ is isomorphic to a basic interval in the lattice of T_1-topologies on some set. Consequently, we restrict ourselves to intervals in Λ in Sections 2–4. In the second section we will examine a subclass of basic intervals in Λ, the intervals of the form $[\mathfrak{J} \cap \mathfrak{J}(a, \mathfrak{U}), \mathfrak{J}]$, where $\mathfrak{J}(a, \mathfrak{U})$ is an ultratopology in Λ. In the third section we explore the relationship between basic intervals and linear intervals in Λ. Finally, we use our results concerning basic intervals to prove that if a T_2 first -countable topology is properly contained between two topologies, then there must be at least $2^{2^{\aleph_0}}$ distinct topologies contained between the two.

The topological definitions of this paper may be found in *Topological Structures* by Thron, while the lattice terminology which is not defined in the paper may be found in *Introduction to Lattice Theory* by Szász. We would like to thank the referee for his helpful suggestions.

1. **Basic intervals of Σ.** Throughout this paper we will use $[\mathcal{S}, \mathfrak{J}]$ to represent the interval $\{\mathcal{R} \in \Sigma \mid \mathcal{S} \leq \mathcal{R} \leq \mathfrak{J}\}$. When writing $[\mathcal{S}, \mathfrak{J}]$ we imply that $\mathcal{S}, \mathfrak{J} \in \Sigma$ and that $\mathcal{S} \leq \mathfrak{J}$.

Definition 1. An interval $[\mathcal{S}, \mathfrak{J}]$ of Σ is called *basic* iff there exists some $a \in X$ such that for every $G \in \mathfrak{J}$, $G - \{a\} \in \mathcal{S}$.

By $G - \{a\}$ we simply mean the set $G \cap (X - \{a\})$ and do not necessarily imply that $a \in G$.

Lemma 1. *If $[\mathcal{S}, \mathfrak{J}]$ is a basic interval of Σ such that $\mathcal{S} \neq \mathfrak{J}$, then there exists a unique $a \in X$ such that for every $G \in \mathfrak{J}$, $G - \{a\} \in \mathcal{S}$.*

Received December 16, 1971. Revision received March 31, 1972.