THE DOUBLE COMMUTANTS OF INVERTIBLY WEIGHTED SHIFTS

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1. Introduction. Let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded operators on the Hilbert space $\mathcal{H}$. For each operator T in $\mathcal{L}(\mathcal{H})$ let $\mathfrak{A}(T)$ be the closure in the weak operator topology of the polynomials in T. If for any subalgebra $\mathfrak{B}$ of $\mathcal{L}(\mathcal{H})$ we set $\mathfrak{B}' = \{T \text{ in } \mathcal{L}(\mathcal{H}): BT = TB \text{ for all } B \text{ in } \mathfrak{B}\}$, then for a large class of operators T, $(\mathfrak{A}(T))'' = \mathfrak{A}(T)$. (For instance, if $\mathcal{H}$ is finite dimensional, this holds for all T.) There is not as yet however a complete description of the class (dc) of operators T satisfying the double commutant condition $(\mathfrak{A}(T))'' = \mathfrak{A}(T)$. A large collection of operators in the class (dc) is given in [4] along with a general discussion of such operators. In this paper we examine the double commutativity of operator weighted shifts. Particular attention is paid to the direct sum of scalar weighted shifts. In this case we completely characterize the double commutant. It is well known that a scalar weighted shift (with nonzero weights) generates a maximal abelian algebra and hence is in the class (dc). We show that there exist shifts with invertible operator weights which are not in the class (dc).

1. Preliminaries. Let $\mathcal{H}$ be a complex Hilbert space and let $l_2(\mathcal{H})$ be the Hilbert space of all square summable sequences in $\mathcal{H}$. Then $IWl_2(\mathcal{H})$ is the class of all operators S on $l_2(\mathcal{H})$ whose action is given by $S\langle x_0, x_1, \cdots\rangle = \langle 0, A_1x_0, A_2x_1, \cdots\rangle$, where $\{A_1, A_2, \cdots\}$ is a uniformly bounded sequence of invertible operators on $\mathcal{H}$. Let $S_0 = I$, the identity operator on $\mathcal{H}$, and let $S_n = A_nA_{n-1} \cdots A_1$. For an operator T on a Hilbert space $\mathcal{K}$ define $\mathfrak{A}(T)$ to be the weakly closed algebra generated by I and T, $\mathfrak{A}'(T)$ to be the commutant of T, and $\mathfrak{A}''(T)$ to be the double commutant of T, that is, $\mathfrak{A}''(T)$ is the commutant of $\mathfrak{A}'(T)$.

LEMMA 2.1. [1; Lemma 2.2] *Let S be the operator in $IWl_2(\mathcal{H})$ with weight sequence $\{A_n\}$. Then an operator B on $l_2(\mathcal{H})$ is in $\mathfrak{A}'(S)$ if and only if B has matrix $[B_{ij}]_{i,j=0}^{\infty}$ relative to the decomposition $l_2(\mathcal{H}) = \mathcal{H} \oplus \mathcal{H} \oplus \cdots$, where $B_{ij} = 0$ for $i < j$ and $B_{ij} = S_iS_{i-j}^{-1}B_{i-j0}S_j^{-1}$ for $i \geq j$.*

LEMMA 2.2. [2; Theorem 3.1] *An operator B on $l_2(\mathcal{H})$ is in $\mathfrak{A}(S)$ if and only if it has matrix as in Lemma 2.1 and for $k \geq 0$, $B_{k0} = \lambda_kS_k$, where $\{\lambda_k\}$ is a sequence of scalars.*

We say that an operator T is in the class (dc) if $\mathfrak{A}''(T) = \mathfrak{A}(T)$. It is shown in [4] that every scalar weighted shift on a separable Hilbert space is in the

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