ON COMMUTATIVE RINGS OF FINITE RANK

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Let R be a commutative ring and let k be a positive integer. Following I. S. Cohen [6] we say that *R has rank k* if each ideal of R has a basis of k elements (and hence R has rank s for each integer $s > k$). Cohen establishes in [6; Theorems 9 and 10] the following result.

THEOREM 1. *If R is an integral domain with identity, then R has finite rank if and only if R is Noetherian and there is a positive integer t such that each localization R_P of R with respect to a maximal ideal P has rank t. More specifically, if R has rank k, then each R_P has rank k; if R is Noetherian and if each R_P has rank t, then R has rank $t + 1$. If R is a local domain, then R has finite rank if and only if the dimension of R is less than 2.*

Since the appearance of [6] numerous variations and generalizations of Theorem 1 have been considered. For example, rings of rank 2 were considered by Matlis [10], Berger [2], Bass [1], and again by Matlis in [11]. Finite generating sets and minimal generating sets for ideals of a ring or for a module have been investigated by Serre [12], Forster [7], Swan [13], [14], Gilmer and Heinzer [9], and Geramita [8]. Brameret [3], [4] and [5] defined and developed some properties of the concept of the width of a module; Wichman continued this study in [15]. In this process of extension and generalization the problem of characterizing Noetherian rings with zero divisors and with finite rank seems to have been passed over. It is the purpose of this paper to provide two such characterizations, Theorems 2 and 3.

Let M be a module over a commutative ring R and let k be a positive integer. If each submodule of M has a basis of k elements, then we say that *M has rank k*. If R is considered as an R-module, this definition agrees with our previous definition; it does not agree with the definition of the same term given by Cohen in [6; 40]. We list some easy consequences of our definition.

PROPOSITION 1. *Let M be a module over the commutative ring R and let N be a submodule of M.*

(1) *If M has rank k, then N and M/N have rank k.*
(2) *If N has rank r and if M/N has rank s, then M has rank $r + s$.*
(3) *If M is the direct sum of the finite family $\{M_i\}_{i=1}^n$ of submodules of M, then M has finite rank if and only if each M_i has finite rank.*

Received February 5, 1972. Revision received March 2, 1972. The author received partial support from National Science Foundation Grant GP-19406.