ASYMPTOTIC VALUES OF FINITELY VALENT FUNCTIONS

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1. Introduction. A function f defined in a domain D is said to be n-*valent* in D if $f(z) - w_0$ has at most n zeros in D for each complex number w_0.

MacLane's class $\mathcal{A}$ [6; §3] is the class of nonconstant holomorphic functions in the unit disc that have asymptotic values at a dense subset of the unit circle. MacLane [6] gave several conditions for a function to belong to class $\mathcal{A}$. The main purpose of this paper is to establish the following sufficient condition (see Theorem 3) for a function to belong to $\mathcal{A}$. A nonconstant holomorphic function f in the unit disc is in class $\mathcal{A}$ if for some positive integer n and for a positive number r_0, the function f is n-valent in each component of the set $\{z : |f(z)| > r_0\}$. Furthermore (see Corollary 3) the set of points at which f has finite linearly accessible asymptotic values is a dense subset of $|z| = 1$.

Theorems 1 and 2 deal with the effect of n-valence on asymptotic tracts. We show that if f is n-valent in the domains forming an asymptotic tract, then the asymptotic tract must be a point tract. Also, if f is finitely valent in the domains forming each asymptotic tract for infinity, then at most one point tract can end at any given point of the unit circle. In Corollary 2 we show that $f \in \mathcal{L}$ (see the definition given below) if there is a sequence $\{r_n\}$ of positive numbers converging to infinity such that each of the level sets $\{z : |f(z)| = r_n\}$ ends at points of $|z| = 1$.

2. Definitions and notation. Let S be a nonempty subset of $|z| < 1$. For each r, $0 < r < 1$, let the components of $S \cap \{z : r < |z| < 1\}$ be $S_j(r)$, $j \in J$. Let $d_j(r)$ be the diameter of $S_j(r)$ and let $d(r) = \sup_{j \in J} d_j(r)$. Clearly d is a nonincreasing function of r. The set S *ends at points* of $|z| = 1$ if $d(r) \downarrow 0$ as $r \uparrow 1$.

A nonconstant holomorphic function f in $|z| < 1$ belongs to the class $\mathcal{L}$ if the set $\{z : |f(z)| = r\}$ ends at points of $|z| = 1$ for each $r > 0$.

MACLANE'S THEOREM. $\mathcal{A} = \mathcal{L}$.

We shall use this result in the proof of Theorem 3.

Let f be a nonconstant holomorphic function in $|z| < 1$. An *asymptotic tract* $\{D(r), c\}$ for the finite asymptotic value c is a set of nonempty domains $D(r)$, one for each $r > 0$, such that

(i) $D(r)$ is a component of $\{z : |f(z) - c| < r\}$

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