ENDOMORPHISMS OF FINITE RANK FLAT MODULES

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In this note we answer a question of W. V. Vasconcelos [8] by characterizing those rings for which surjective endomorphisms of finite rank flat modules are always injective. We also give an example of a flat R-module of finite exterior rank admitting a surjective endomorphism which is not injective.

1. Let R be a commutative unitary ring and let M be an R-module. The rank of M at a prime ideal p of R, denoted $rk_M(p)$, is defined as the dimension of $M \otimes_R k(p)$ considered as a vector space over $k(p) = R_p/p_p$. We say that M has finite rank if $rk_M(p)$ is finite for every prime ideal p of R and we say M has bounded rank if for some integer n $rk_M(p) \leq n$ for every $p \in \text{Spec}(R)$. The exterior rank of M, denoted ext rank (M), is the supremum of the set of nonnegative integers n such that $\wedge^n M \neq 0$. Since for every nonnegative integer n and for every prime ideal p of R the natural map $(\wedge^n M) \otimes_R R_p \rightarrow \wedge^n (M \otimes_R k(p))$ is surjective, it follows that if M has finite rank, then $\sup \{rk_M(p) \mid p \in \text{Spec}(R)\} \leq \text{ext rank}(M)$. That this inequality may be strict follows from our main theorem. Following Bass [1] we say an ideal N of a (not necessarily commutative) ring R is left T-nilpotent if given any sequence $a_1, a_2, \cdots$ of elements of N, there is an integer n such that $a_1 a_2 \cdots a_n = 0$. We use the following lemma.

LEMMA. *Let $f: R \rightarrow S$ be a unitary ring homomorphism. The following are equivalent.*

(1) *If M is a flat left R-module such that $S \otimes M = 0$, then $M = 0$.*
(2) Ker (f) *is left T-nilpotent.*
(3) *If $h: M \rightarrow N$ is a surjective homomorphism of flat left R-modules such that $1_S \otimes h$ is injective, then h is injective.*

Proof. (1) $\Rightarrow$ (2). Let $a_1, a_2, \cdots$ be a sequence of elements of Ker (f). Let $h_i : R \rightarrow R$ be the homomorphism of left R-modules defined by $h_i(x) = xa_i$ and let M be the direct limit of the system $R \rightarrow^{h_1} R \rightarrow^{h_2} R \rightarrow \cdots$. M is flat since it is the direct limit of free R-modules. Since $1_S \otimes h_i = 0$ for all i and since $S \otimes$—commutes with direct limits, then $S \otimes M = 0$. Hence by (1) $M = 0$. Thus there is an integer n such that $0 = h_n h_{n-1} \cdots h_1(1) = a_1 a_2 \cdots a_n$.

(2) $\Rightarrow$ (3). Let $0 \rightarrow L \rightarrow M \rightarrow^h N \rightarrow 0$ be an exact sequence of left R-modules with $1_S \otimes h$ injective and with M and N flat. Then L is flat [2; 31, Proposition 5] and the sequence remains exact when $S \otimes$—is applied; hence $S \otimes_R L = 0$. Since L is flat, tensoring with L preserves the exact sequence $0 \rightarrow$ Ker $(f) \rightarrow$

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