MATRIX FIELDS OVER PRIME FIELDS

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1. Introduction and notation. Let R be a ring with identity and let $(R)_n$ denote the complete matrix ring of all $n \times n$ matrices over R under normal matrix addition and multiplication. Let M be a subring of $(R)_n$. Then M is called a *matrix field* of $(R)_n$, or simply a *matrix field*, if and only if M is itself a field. Although it is not standard, we find it convenient to refer to M as a subfield of the ring $(R)_n$. We are interested in characterizing all subfields of $(R)_n$ and, whenever appropriate, in determining the number of distinct subfields of $(R)_n$. The author has succeeded in characterizing all subfields of $(F)_n$, where the field F is a finite extension of its prime subfield F_p, and has enumerated the distinct subfields of $(GF(q))_n$. The results given here are motivated by taking R first as the ring generated by the identity of an arbitrary integral domain D and then by taking R as the quotient field of this ring. In addition to characterizing all subfields of $(Z)_n$, $(Q)_n$ and $(GF(p))_n$, we give constructive techniques for extending matrix fields within these rings.

We emphasize that in all cases we are concerned with finding all subrings of $(R)_n$ which are fields and not merely those having the identity I_n of $(R)_n$ as their own identity. For example, it is easily verified that the subring M of $(Q)_2$ given by

$$M = \left\{ x \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} : x \in Q \right\}$$

is a subfield of $(Q)_2$ and has the matrix $\begin{vmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix}$ as its identity.

2. Subfields of $(Z)_n$. In this section we consider subfields of $(R)_n$ where R is the ring generated by the identity of an integral domain of characteristic zero. The result over Z is thus a special case of the following theorem.

THEOREM 1. *Let D be an integral domain having characteristic zero and let R be the subring generated by the identity of D. Then $(R)_n$ has no subfields.*

Proof. Let F be the quotient field of D and consider R as imbedded in F. Suppose $(R)_n$ has a subfield, say M. Then M is a subfield of $(F)_n$. Since M is a field, then M has an identity, call it I. There are two cases to consider.

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