FRÉCHET DISTANCE AND THE UNIFORM CONVERGENCE OF QUASICONFORMAL MAPPINGS

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1. Introduction. It is the purpose of this paper to generalize a classical theorem of Radó [7] concerning the uniform convergence of conformal mappings of the unit disk in the plane to a theorem on the uniform convergence of quasiconformal mappings in n-space. The question of uniform convergence of conformal and quasiconformal maps in the plane has been studied from various points of view by Radó, Courant [1], Markouchevitch [4], Gaier [2] and Wilson [9], and many of their results remain valid in n-space. (See [6].) Our specific interest here is the result of Radó.

2. Notation and terminology. We denote by R^n the n-dimensional Euclidean space and by $\bar{R}^n$, for $n \geq 2$, its one point compactification, $R^n \cup \{\infty\}$. If $x \in R^n$, x_i, $i = 1, 2, \cdots, n$, will be the i-th coordinate of x with respect to a fixed orthonormal basis $\{e_1, \cdots, e_n\}$. Stereographic projection from the n-sphere induces a natural metric q on $\bar{R}^n$, the chordal metric, and all topological considerations in this paper refer to $\bar{R}^n$ and the topology induced on it by q. For a subset A of $\bar{R}^n$ we denote by $\bar{A}$, int A, $C(A)$, ∂A and $q(A)$ the closure, interior, complement, boundary and chordal diameter of A respectively. If A and B are subsets of $\bar{R}^n$, $A \backslash B$ is the difference set $A \cap C(B)$ and $q(A, B)$ denotes the chordal distance between A and B. If $x \in R^n$ and $r > 0$, $B^n(x, r)$ is the open (Euclidean) ball of radius r with center at x. A domain in $\bar{R}^n$ is a nonempty, open, connected subset of $\bar{R}^n$. By a continuum is meant a closed, connected set containing at least two points.

By a path in $\bar{R}^n$ we understand a continuous mapping of a closed interval into $\bar{R}^n$. If E, F and G are subsets of $\bar{R}^n$, the notation $\Delta(E, F:G)$ is used for the family of all paths joining E and F in G, i.e., a path $\gamma:[a, b] \to \bar{R}^n$ belongs to $\Delta(E, F:G)$ if and only if one endpoint belongs to E, one endpoint belongs to F and $\gamma(t) \in G$ for $a < t < b$. If Γ is a family of paths in $\bar{R}^n$, $F(\Gamma)$ will be the set of all nonnegative, extended real valued, Borel measurable functions ρ on $\bar{R}^n$ such that

$$\int_\gamma \rho \, ds \geq 1$$

for each rectifiable $\gamma \in \Gamma$. The n-modulus of Γ, written $M(\Gamma)$, is defined by

$$M(\Gamma) = \inf_{F(\Gamma)} \int_{R^n} \rho^n \, dx$$

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