FRÉCHET DISTANCE AND THE UNIFORM CONVERGENCE OF QUASICONFORMAL MAPPINGS

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1. Introduction. It is the purpose of this paper to generalize a classical theorem of Radó [7] concerning the uniform convergence of conformal mappings of the unit disk in the plane to a theorem on the uniform convergence of quasiconformal mappings in n-space. The question of uniform convergence of conformal and quasiconformal maps in the plane has been studied from various points of view by Radó, Courant [1], Markouchevitch [4], Gaier [2] and Wilson [9], and many of their results remain valid in n-space. (See [6].) Our specific interest here is the result of Radó.

2. Notation and terminology. We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space and by $\mathbb{R}_n$, for $n \geq 2$, its one point compactification, $\mathbb{R}_n \cup \{\infty\}$. If $x \in \mathbb{R}^n$, $x_i$, $i = 1, 2, \ldots, n$, will be the $i$-th coordinate of $x$ with respect to a fixed orthonormal basis $\{e_1, \ldots, e_n\}$. Stereographic projection from the n-sphere induces a natural metric $q$ on $\mathbb{R}^n$, the chordal metric, and all topological considerations in this paper refer to $\mathbb{R}^n$ and the topology induced on it by $q$. For a subset $A$ of $\mathbb{R}^n$ we denote by $\overline{A}$, int $A$, $C(A)$, $\partial A$ and $q(A)$ the closure, interior, complement, boundary and chordal diameter of $A$, respectively. If $A$ and $B$ are subsets of $\mathbb{R}^n$, $A \setminus B$ is the difference set $A \cap C(B)$ and $q(A, B)$ denotes the chordal distance between $A$ and $B$. If $x \in \mathbb{R}^n$ and $r > 0$, $B(x, r)$ is the open (Euclidean) ball of radius $r$ with center at $x$. A domain in $\mathbb{R}^n$ is a nonempty, open, connected subset of $\mathbb{R}^n$. By a continuum is meant a closed, connected set containing at least two points.

By a path in $\mathbb{R}^n$ we understand a continuous mapping of a closed interval into $\mathbb{R}^n$. If $E$, $F$ and $G$ are subsets of $\mathbb{R}^n$, the notation $\Delta(E, F; G)$ is used for the family of all paths joining $E$ and $F$ in $G$, i.e., a path $\gamma:[a, b] \to \mathbb{R}^n$ belongs to $\Delta(E, F; G)$ if and only if one endpoint belongs to $E$, one endpoint belongs to $F$ and $\gamma(t) \in G$ for $a < t < b$. If $\Gamma$ is a family of paths in $\mathbb{R}^n$, $F(\Gamma)$ will be the set of all nonnegative, extended real valued, Borel measurable functions $\rho$ on $\mathbb{R}^n$ such that

$$\int_{\gamma} \rho \, ds \geq 1$$

for each rectifiable $\gamma \in \Gamma$. The $n$-modulus of $\Gamma$, written $M(\Gamma)$, is defined by

$$M(\Gamma) = \inf_{F(\Gamma)} \int_{\mathbb{R}^n} \rho^n \, dx$$

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