ENUMERATION OF SEQUENCES BY RISES AND FALLS: A REFINEMENT OF THE SIMON NEWCOMB PROBLEM

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1. Introduction. Let $e_1, e_2, \cdots, e_n$ be positive integers and consider sequences

$$\sigma = (a_1, a_2, \cdots, a_N) \tag{1.1}$$

of length

$$N = e_1 + e_2 + \cdots + e_n, \tag{1.2}$$

where

$$a_i \in \{1, 2, \cdots, n\} \qquad i = 1, 2, \cdots, N \tag{1.3}$$

and each element j in σ occurs exactly e_j times. We shall call σ a sequence of *specification* $[e_1, \cdots, e_n]$. A pair of consecutive elements a_i, a_{i+1} is called a *rise* if $a_i < a_{i+1}$, a *fall* if $a_i > a_{i+1}$, a *level* if $a_i = a_{i+1}$; also it is sometimes convenient to count a conventional rise to the left of a_1 and a conventional fall to the right of a_N. If r, s and t denote the number of rises, falls and levels in σ respectively, then it is clear that

$$r + s + t = N + 1. \tag{1.4}$$

Simon Newcomb's problem [4; Chapter 8] is to determine $A(e_1, \cdots, e_n \mid r)$, the number of sequences (1.1) with exactly r rises. The simplest case is that in which $s_1 = \cdots = s_n = 1$. The solution is the Eulerian number $A_{n,r}$ defined by

$$\frac{1-y}{e^{x(y-1)} - y} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{r=1}^{n} A_{n,r} y^{n-r} \tag{1.5}$$

which implies

$$A_{n,r} = \sum_{j=0}^{r} (-1)^j \binom{n+1}{j} (r - j + 1)^n. \tag{1.6}$$

Dillon and Roselle [3] have recently solved the general Simon Newcomb problem. They show that the solution is an *extended* Eulerian number as defined by the author [1]. Put

$$\frac{1-\lambda}{\zeta(s) - \lambda} = \sum_{m=1}^{\infty} m^{-s} (\lambda - 1)^{-\Omega(m)} \sum_{r=1}^{\Omega(m)} A(m, r) \lambda^{\Omega(m)-r}, \tag{1.7}$$

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