ON STOCHASTIC MATRICES WHOSE ABSOLUTE SMALLEST CHARACTERISTIC ROOT IS REAL

ALFRED BRAUER AND IVEY C. GENTRY

Let A be a generalized stochastic matrix with row sum s. It is well known that s is the trivial root of A. But the nontrivial roots can be very different. It was proved by the first author that the second largest root of A may be positive too [4]. In a paper to be published soon we proved that all the nontrivial roots can be imaginary, namely, if A is the matrix of a tied tournament [6]. We will show in this paper that under certain conditions the absolute smallest root is real. We will prove the following theorem.

THEOREM. *Let a_{11} and a_{22} be the two smallest main diagonal elements of A. Assume that*

$$a_{22} > a_{11}(3 - 2\sqrt{2}) + s(2\sqrt{2} - 2). \tag{1}$$

Then the smallest root of A is real.

Proof. It was proved by the first author [1] (see also [5]) that all the characteristic roots lie in the interior or on the boundary of the oval of Cassini

$$\{(x - a_{11})^2 + y^2\}\{(x - a_{22})^2 + y^2\} = (s - a_{11})^2(s - a_{22})^2. \tag{2}$$

The vertices of this oval are (see [3])

$$x = \tfrac{1}{2}(a_{11} + a_{22}) \pm \tfrac{1}{2}\{(a_{11} - a_{22})^2 \pm 4(s - a_{11})(s - a_{22})\}^{\frac{1}{2}}. \tag{3}$$

This oval is doubly connected if

$$(a_{11} - a_{22})^2 > 4(s - a_{11})(s - a_{22}).$$

Hence

$$a_{11}^2 - 2a_{11}a_{22} + a_{22}^2 > 4s^2 - 4s(a_{11} + a_{22}) + 4a_{11}a_{22}$$

$$a_{22}^2 - a_{22}(6a_{11} - 4s) + a_{11}^2 - 4s^2 + 4sa_{11} > 0$$

$$a_{22} > 3a_{11} - 2s + (4s^2 - 12a_{11}s + 9a_{11}^2 - a_{11}^2 + 4s^2 - 4sa_{11})^{\frac{1}{2}}$$

$$a_{22} > 3a_{11} - 2s + (8s^2 - 16a_{11}s + 8a_{11}^2)^{\frac{1}{2}}$$

$$a_{22} > 3a_{11} - 2s + 2\sqrt{2}(s - a_{11}) = a_{11}(3 - 2\sqrt{2}) + s(2\sqrt{2} - 2).$$

If (1) holds, then neither a_{22}, nor a_{33}, $\cdots$, a_{nn} are points of that branch of (2) containing the point a_{11}.

Let M_ν be the union of the simply connected regions of all the $\frac{1}{2}n(n-1)$ ovals containing the point $a_{\nu\nu}$. Since M_1 contains a_{11} and no other point a_{kk},

Received December 31, 1971.