A UNIVERSAL COUNTABLE FIRST-COUNTABLE HAUSDORFF SPACE

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In the present paper we construct a countable, first-countable, connected Hausdorff space which contains a homeomorphic copy of every countable first-countable Hausdorff space.

Perhaps the best known example of a connected countable Hausdorff space was given by Bing [1].

1. The A-construction. If S is any set, let S^+ be the set whose elements are those of S together with one other element which we call 0. (To avoid logical difficulties we may always choose 0 to be $\{S\}$.) Give S^+ the topology where if $T \subset S^+$, T is open if and only if $T = S^+$ or $0 \notin T$. Now if X is any topological space, define AX to be $X \times |X|^+$, where $|X|$ is the underlying set of X.

We inductively define finite topological spaces A^k for all integers $k \geq 1$ as follows. A^1 is any one-point space and $A^{k+1} = AA^k$ for all $k \geq 1$. Let $i_k : A^k \to A^{k+1}$ be the function where $i_k a = (a, a)$ for all $a \in A^k$, and let $p^k : A^{k+1} \to A^k$ be the map (a map is a continuous function) where p^k is the projection onto the first factor. Thus $p^k i_k$ is the identity on A^k. Let A^∞ be a direct limit (in the category of sets and one-to-one functions) of the A^k; that is, we have a one-to-one function $I_k : A^k \to A^\infty$ for all k such that

(i) $I_{k+1} i_k = I_k$ for all $k \geq 1$

(ii) if $a \in A^\infty$, $a \in \operatorname{Im} I_k$ for some k.

We see by (ii) that A^∞ is countable. For each $k \geq 1$ let $P^k : A^\infty \to A^k$ be defined as follows. If $a \in A^\infty$ and if $a = I_r b$ for some $r \geq k$ and some $b \in A^r$, let $p^k a = p_r^k b$, where $p_r^k = p^k p^{k+1} \cdots p^{r-1}$. Since $I_{r+1} i_r = I_r$ and $p_{r+1}^k i_r = p_r^k$, p^k is well-defined. Furthermore, the following diagram is commutative for all k.

$$\begin{tikzcd} & A^\infty \arrow[dl, "P^k"'] \arrow[dr, "P^{k+1}"] & \\ A^k & & A^{k+1} \arrow[ll, "p^k"'] \end{tikzcd}$$

Let A^∞ have the coarsest topology such that each P^k is continuous. Since each A^k has only a finite topology, A^∞ has a countable subbase and is therefore first-countable; if $x \neq y \in A^\infty$, then $x = I_k a$ and $y = I_k b$ for some k; $(P^{k+1})^{-1}(A^k \times \{a\})$ and $(P^{k+1})^{-1}(A^k \times \{b\})$ are disjoint open neighborhoods of x and y so A^∞ is Hausdorff.

THEOREM 1. *Every countably infinite first-countable Hausdorff space X can be embedded in A^∞.*

Received March 10, 1970. Revision received July 13, 1970.