MACKEY TOPOLOGIES WHICH ARE LOCALLY CONVEX RIESZ TOPOLOGIES

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1. Introduction. Let L be a Riesz space and Θ a locally convex Hausdorff topology on L. Then Θ is called a *locally convex Riesz topology* if there is a local base at 0 for Θ consisting of solid sets. In this case the (topological) dual space $(L, \Theta)^*$ is an ideal in the order dual $L^\sim$ and $(L, \Theta)^*$ distinguishes points in L.

Let L be a Riesz space and I an ideal in $L^\sim$ which distinguishes points in L. Let $\mathcal{I}$ be the family of all order intervals in I. Then $\{A^\circ : A \in \mathcal{I}\}$ is a local base at 0 for a locally convex Riesz topology $|\sigma|\ (L, I)$ on L and $(L, |\sigma|\ (L, I))^* = I$. Let $\mathcal{S}$ be the family of all $\sigma(I, L)$-compact solid absolutely convex sets in I. Then $\{S^\circ : S \in \mathcal{S}\}$ is a local base at 0 for a locally convex Riesz topology $|\tau|\ (L, I)$ and $(L, |\tau|\ (L, I))^* = I$. Indeed, the following analogue to the Mackey–Arens Theorem is well-known. If Θ is a locally convex Riesz topology on L, then $(L, \Theta)^* = I$ if and only if $|\sigma|\ (L, I) \leq \Theta \leq |\tau|\ (L, I)$.

Again suppose I is an ideal in $L^\sim$ such that I distinguishes points in L. A. L. Peressini [6] has shown that the weak topology $\sigma(L, I)$ is a locally convex Riesz topology (i.e., $\sigma(L, I) = |\sigma|\ (L, I)$) if and only if order intervals in I are contained in finite dimensional subspaces of I. A more complex problem is to determine when the Mackey topology $\tau(L, I)$ is a locally convex Riesz topology (i.e., when $\tau(L, I) = |\tau|\ (L, I)$). It is well-known that $\tau(L, L^\sim)$ is a locally convex Riesz topology whenever $L^\sim$ distinguishes points in L. A more profound result is that if L is Dedekind complete and if the band $L_n^\sim$ of normal integrals in $L^\sim$ distinguishes points in L, then $\tau(L, L_n^\sim)$ is a locally convex Riesz topology. This was shown in 1960 by I. Amemiya [1] and again in 1967 by D. H. Fremlin [2].

If I distinguishes points in L, then a necessary and sufficient condition that $\tau(L, I)$ be a locally convex Riesz topology is that every $\sigma(I, L)$-compact absolutely convex subset of I be contained in a solid $\sigma(I, L)$-compact absolutely convex subset of I (Lemma 2.1). Fremlin shows that if L is Dedekind complete and $L_n^\sim$ distinguishes points in L, then every $\sigma(L_n^\sim, L)$-compact set is contained in a solid $\sigma(L_n^\sim, L)$-compact absolutely convex set. In §3 we show that the condition that L be Dedekind complete in Fremlin's Theorem can be weakened to either that L have the projection property (Theorem 3.5) or that L be Dedekind σ-complete (Theorem 3.8). However, we give examples in §4 to show that if L is a Riesz space such that $L_n^\sim$ distinguishes points in L, then the principal

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