DILATATIONS OF QUASICONFORMAL BOUNDARY CORRESPONDENCES

BY F. W. GEHRING

1. Introduction. Suppose that $n \geq 2$ and that Γ is a family of curves in $\bar{R}^n$, the one-point compactification of Euclidean n-space R^n. For $p \in [1, \infty)$ the *p-modulus* of Γ is defined by

$$M_p(\Gamma) = \inf \int_{R^n} \rho^p \, dm_n , \tag{1}$$

where the infimum is taken over the family $F(\Gamma)$ of Borel measurable functions $\rho: R^n \to [0, \infty]$ such that $\int_\gamma \rho \, ds \geq 1$ for each locally rectifiable curve $\gamma \in \Gamma$. Suppose next that D and D' are domains in $\bar{R}^n$ and that $f: D \to D'$ is a homeomorphism. We call

$$K_I(f) = \sup \frac{M_n(f[\Gamma])}{M_n(\Gamma)} ,$$

$$K_O(f) = \sup \frac{M_n(\Gamma)}{M_n(f[\Gamma])} ,$$

$$K(f) = \max (K_I(f), K_O(f))$$

the *inner*, *outer* and *maximal dilatations* of f, where the suprema are taken over all curve families Γ in D for which $M_n(\Gamma)$ and $M_n(f[\Gamma])$ are not simultaneously 0 or ∞. Then

$$1 \leq K_I(f) \leq K_O(f)^{n-1}, \qquad 1 \leq K_O(f) \leq K_I(f)^{n-1}, \tag{2}$$

and $K_I(f) = K_O(f) = K(f)$ when $n = 2$. (See, for example, [8; 34.5].) We say that f is *K-quasiconformal* if $K(f) \leq K < \infty$ and that f is *quasiconformal* if $K(f) < \infty$.

Now suppose that D is a half-space in R^n and that $f: D \to D$ is an n-dimensional quasiconformal mapping. Then f has a homeomorphic extension to $\bar{D}$. Moreover when $n \geq 3$ the induced boundary mapping $g: \partial D \to \partial D$ is itself an $(n - 1)$-dimensional quasiconformal mapping ([2] and [7]). Interesting consequences of this fact are given in [7]. In [5] it was shown that

$$K(g) \leq \min (K_I(f), K_O(f)) \tag{3}$$

when $n = 3$, and this inequality was used in [3], [4] and [5] to study extremal quasiconformal mappings in R^3. In this paper we show that

$$K_I(g) \leq K_I(f) \quad \text{and} \quad K_O(g) \leq K_O(f) \tag{4}$$

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