THE DISTRIBUTION OF k-TH POWER RESIDUES AND NON-RESIDUES IN THE INTEGRAL DOMAIN $Z(\sqrt{-2})$

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1. Introduction and notation. In 1968 Jordan [7] lifted the results of Burgess [1] and Davenport and Erdös [4] to the integral domain of the Gaussian integers by establishing the following two theorems.

THEOREM 1.1. *If α is a quadratic non-residue modulo the Gaussian prime γ and $|\alpha| \leq |\beta|$ for β a quadratic non-residue modulo γ, then $|\alpha| < |\gamma|^{a+\epsilon}$ for all $\epsilon > 0$, for $a = (4\sqrt{e})^{-1}$, and for all sufficiently large $|\gamma|$'s, where e is the base for the natural logarithms.*

THEOREM 1.2. *Let $k \mid (|\gamma|^2 - 1)$, where γ is a Gaussian prime. Let $(4a)^{-1}$ be the unique solution of $\Gamma(x) = k^{-1}$, where $\Gamma(x)$ is the Dickman–de Bruijn function. If α is a k-th power non-residue modulo γ and $|\alpha| \leq |\beta|$ for β a k-th power non-residue modulo γ, then $|\alpha| < |\gamma|^{a+\epsilon}$ for all $\epsilon > 0$ and for all sufficiently large $|\gamma|$'s.*

The purpose of this paper is to establish the results of Theorem 1.1 and Theorem 1.2 in the Euclidean domain $Z(\sqrt{-2})$.

Throughout the remainder of this paper the Greek letters α, β, μ and σ will represent integers in $Z(\sqrt{-2})$, where Z represents the class of rational integers. The Greek letters τ and ξ will be complex numbers while the Greek letters ρ and ρ_i will always denote primes in $Z(\sqrt{-2})$. The Latin letters j, k, n, r, t, w, h_i and n_i will represent rational integers and the Latin letters a, b, y_i and c_i will represent real constants. The Latin letters q_i , q and p will represent rational primes while e always represents the base for the natural logarithms and i is the imaginary unit. We will represent the complex plane by C.

As in Jordan [7] we will assume that α is a k-th power residue modulo ρ iff $\mu^k \equiv \alpha (\text{mod } \rho)$ is solvable in $Z(\sqrt{-2})$. Otherwise α is called a k-th power non-residue modulo ρ.

2. Lemmas. We will illustrate the integers in $Z(\sqrt{-2})$ by lattice points in a Cartesian coordinate system, where the horizontal grid lines are $\sqrt{2}$ units apart and the vertical grid lines are 1 unit apart. For any μ let $\tilde{\mu}$ be the set of integers in $Z(\sqrt{-2})$ inside the rectangle whose vertices are $(\pm 1 \pm \sqrt{2}i)\mu/2$ or on the portion of the boundary given by the half-open line segments $(\pm(-1 + \sqrt{2}i)\mu/2, (-1 - \sqrt{2}i)\mu/2]$.

In [11] we find the following lemma.

LEMMA 2.1. *The set $\tilde{\mu}$ is a complete residue system modulo μ and the cardinality of $\tilde{\mu}$ is $|\mu|^2$.*

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