GENERATING FUNCTIONS FOR POWERS OF THIRD ORDER RECURRENCE SEQUENCES

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1. Introduction. Generating functions for powers of certain second order recurrence sequences have been investigated in this journal by Riordan [6], Carlitz [1] and Horadam [3].

Our object here is to find a formula (obtained in (2.2)) for $k_r(x)$, where

$$k_r(x) = \sum_{n=0}^{\infty} k_n^r x^n \qquad (r \geq 1)$$

and k_n satisfies the third order recurrence relation

$$k_n = Pk_{n-1} + Qk_{n-2} + Rk_{n-3} \; (n \geq 3) \tag{1.1}$$

with suitable initial values k_0, k_1, k_2 and where P, Q, R are arbitrary integers.

Relation (1.1) has an auxiliary equation $x^3 - Px^2 - Qx - R = 0$, which we suppose has three distinct, real roots given by α, β, γ. Write $p = \alpha + \beta$, $q = \alpha\beta$. Now (1.1) can be expressed as

$$(E^2 - pE + q)w_n = 0, \tag{1.2}$$

where E is an operator defined by $Ek_n = k_{n+1}$ and where we have replaced $(E - \gamma)k_n$ by w_n. Suppose further that $w_0 = a$ and $w_1 = b$ so that $\{w_n\}$ represents the generalized sequence of numbers studied in detail in [4] and [5].

2. Some generating functions. From the above we get

$$\begin{aligned} k_{n+1}^r &= (w_n + \gamma k_n)^r \\ &= w_n^r + \gamma^r k_n^r + \sum_{j=1}^{r-1} \binom{r}{j} w_n^j \gamma^{r-j} k_n^{r-j}. \end{aligned}$$

Thus

$$\sum_{n=0}^{\infty} k_{n+1}^r x^{n+1} = x \sum_{n=0}^{\infty} w_n^r x^n + \gamma^r x \sum_{n=0}^{\infty} k_n^r x^n + x \sum_{n=0}^{\infty} \sum_{j=1}^{r-1} \binom{r}{j} w_n^j \gamma^{r-j} k_n^{r-j} x^n$$

and

$$(1 - \gamma^r x)k_r(x) = k_0^r + xw_r(x) + x \sum_{n=0}^{\infty} \sum_{j=1}^{r-1} \binom{r}{j} w_n^j \gamma^{r-j} k_n^{r-j} x^n. \tag{2.1}$$

Lemma.

$$\sum_{n=0}^{\infty} w_n^j k_n^{r-j} x^n = \sum_{i=0}^{j} \binom{j}{i} A^{j-i} B^i k_{r-j}(\alpha^{j-i} \beta^i x),$$

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