WEIGHTED QUADRATIC MEANS OF HILBERT TRANSFORMS

By Paul Koosis

Introduction. Let $F(z)$ be an entire function of exponential type. A beautiful and important theorem of Beurling and Malliavin [3] states that if

$$\int_{-\infty}^{\infty} \frac{\log^+ |F(x)|}{1 + x^2}\, dx < \infty,$$

then there are non-zero entire functions of arbitrarily small exponential type whose products with $F(z)$ are bounded on the real axis. The condition on $F(x)$ involved here is, in form, identical to one figuring in the statement of Szegö's theorem [2; 299] and various other results in analysis, all arising ultimately from Jensen's formula. It enters, however, into the work of Beurling and Malliavin in entirely different fashion, and there seems to be no connection between its occurrence there and in the other more classical results. In spite of this, it is natural to believe that the appearances of the same criterion in the context of these different problems must be somehow related.

The work set forth below came out of an attempt to discover such a relation. The idea was to try to obtain the theorem of Beurling and Malliavin from more elementary results like that of Szegö. It should be said at the outset that I have been unable to do this. Indeed, I am not altogether satisfied with the present study, and had almost decided not to publish it. The recent appearance of a paper by Adamian, Arov, and Krein [1], devoted to a different aspect of some of the material considered below, has caused me to change my mind. Some of the results given here are either contained in their paper, or can be easily derived from others found in it. Their methods, however, are based on the theory of extension of operators in Hilbert space, and are hence much more elaborate than the elementary ones I employ. The presentation which follows may thus prove useful to other workers in analysis, especially since some interest in these questions seems now to be developing. I have, incidentally, used an idea from the paper of Adamian, Arov, and Krein, in order to improve one of my own results slightly. This improvement lends a more satisfactory appearance to the work, but is of no practical consequence for the object I originally had in mind.

If one sets out to try to derive the theorem of Beurling and Malliavin from that of Szegö, one is quickly led to a problem involving Hilbert transforms which goes roughly as follows:

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